Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 27, 2003*
Accepted: *January 27, 2004*

Invariant homology on standard model manifolds

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Abstract: Torus-fibered Calabi-Yau threefolds Z, with base $d\mathbb{P}_9$ and fundamental group $\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_2$, are reviewed. It is shown that $Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, where $X = B \times_{\mathbb{P}^1} B'$ are elliptically fibered Calabi-Yau threefolds that admit a freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group. B and B' are rational elliptic surfaces, each with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of automorphisms. It is shown that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant classes of curves of each surface have four generators which produce, via the fiber product, seven $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant generators in $H_4(X, \mathbb{Z})$. All invariant homology classes are computed explicitly. These descend to produce a rank seven homology group $H_4(Z, \mathbb{Z})$ on Z. The existence of these homology classes on Z is essential to the construction of anomaly free, three family standard-like models with suppressed nucleon decay in both weakly and strongly coupled heterotic superstring theory.

Keywords: Superstrings and Heterotic Strings, M-Theory, Superstring Vacua, Differential and Algebraic Geometry.

http://jhep.sissa.it/archive/papers/jhep012004059/jhep012004059.pdf

Contents

1. Introduction

In [1, 2, 3] Hořava and Witten showed that the simplest vacuum of strongly coupled heterotic superstring theory consists of an eleven dimensional bulk space bounded on either end of the eleventh dimension by a ten dimensional orbifold fixed plane. Each fixed plane supports an E_8 supergauge theory on its worldvolume. This theory was compactified on Calabi-Yau threefolds to produce, at low energy, an effective five dimensional bulk space bounded in the fifth dimension by two three-branes. When the Calabi-Yau space supports a non-trivial E_8 gauge instanton with structure group $G \subset E_8$, the gauge group on that three-brane will no longer be E_8. Rather, it is reduced to a group H which is the commutant of G in E_8. In addition to these "end-of-the-world" three-branes, there are, generically, five-branes in the bulk space which are wrapped on a holomorphic curve in the Calabi-Yau threefold. This five-dimensional theory with three-branes is called heterotic M-theory. It is a fundamental paradigm for the so called "brane-world" theories of particle physics. The basic construction of heterotic M-theory was presented in [4, 5, 6].

JHEP01(2004)059

To produce phenomenologically relevant models of particle physics on one of the three-branes, called the "observable brane", it is essential to construct the appropriate gauge instantons on the Calabi-Yau threefold. This is similar to constructing instantons on a flat $\mathbb{R}^4$ manifold, using for example, the ADHM method [7]. Now, however, instantons satisfying the hermitian Yang-Mills equations must be found on a curved Calabi-Yau threefold, a much more challenging problem. The basic ideas for such a construction were presented in [8, 9] for elliptically fibered Calabi-Yau threefolds using the equivalent concept of stable, holomorphic vector bundles. Extending this work, it was shown in [10, 11] that phenomenologically relevant grand unified theories (GUTs) can, indeed, be constructed on the observable brane. This is accomplished by compactifying Hořava-Witten theory on elliptically fibered Calabi-Yau threefolds with trivial fundamental group that support holomorphic vector bundles with structure group $G = \mathrm{SU}(n)$. Choosing n to be 5 and 4, for example, leads to GUT groups H of SU(5) and SO(10) respectively.

Constructing holomorphic vector bundles that break the E_8 gauge theory to standard-like models is more difficult. However, it was shown in [12]–[15] that three-family models with gauge group $H = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ do exist in heterotic M-theory. This was accomplished as follows. First, elliptically fibered Calabi-Yau threefolds with non-trivial fundamental group $\mathbb{Z}_2$ were constructed. It was found to be convenient to choose the base surface of these threefolds to be $B = d\mathbb{P}_9$. See also [16]. Second, new methods were introduced for producing stable, holomorphic vector bundles with structure group $G = \mathrm{SU}(5)$ over these Calabi-Yau spaces. Such instantons break the E_8 group to SU(5), as mentioned above. Now, however, the non-trivial $\mathbb{Z}_2$ fundamental group allows topologically stable flat bundles, that is, $\mathbb{Z}_2$ Wilson lines, which break SU(5) down to $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. Importantly, by sufficiently restricting the Calabi-Yau threefolds, it was shown that new homology classes can appear that allow the theory to be anomaly free and to admit three families of quarks and leptons.

Although very compelling in many ways, standard-like models constructed from $G = \mathrm{SU}(5)$ instantons and $\mathbb{Z}_2$ Wilson lines have a potential problem. It is well-known that superstring vacua associated with GUT theories with gauge group SU(5) may allow nucleon decay more rapid than the experimental bounds. Although the nucleon decay rate can be reduced by imposing certain discrete symmetries and by superstring effects, such vacua do not exhibit a natural mechanism for suppressing nucleon decay. This problem was emphasized by Witten [17, 18] and discussed in [19]. The question of nucleon decay has motivated the authors of this paper to attempt to construct alternative standard-like models of particle physics in heterotic M-theory based on $G = \mathrm{SU}(4)$ instantons and $\mathbb{Z}_2 \times \mathbb{Z}_2$ Wilson lines. As mentioned above, SU(4) instantons break E_8 to an SO(10) GUT group, which is further broken to a standard-like model by a $\mathbb{Z}_2 \times \mathbb{Z}_2$ Wilson line. The key point is that the gauge group of this standard-like model now contains an unbroken $\mathrm{U}(1)_{B-L}$ factor, which naturally suppresses nucleon decay.

This program was begun in [20], where the requisite torus-fibered Calabi-Yau threefolds with fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$ were constructed. Before proceeding with the construction of SU(4) instantons on these manifolds, however, it is essential that one compute the transformation laws of certain elements of the Calabi-Yau homology ring under the $\mathbb{Z}_2 \times \mathbb{Z}_2$

automorphism group. This is necessary in order to establish the existence and properties of homology classes that are invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$. As discussed in [12, 13, 14, 15], such invariant classes are essential to produce three-family, anomaly free standard-like models. This rather technical, but very important, step in our program is carried out in this paper in detail.

To be specific, in section 2 we briefly review the construction of torus-fibered Calabi-Yau threefolds with non-trivial fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$. It is shown that such threefolds have the form $Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, where X is an elliptically fibered Calabi-Yau threefold that is the fiber product of two rational elliptic surfaces B and B'. Section 3 is devoted to a detailed review of the properties of restricted surfaces B that admit at least a $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group. In addition, the relevant involutions $(-1)_B$, α_B, t_{e_6} and t_{e_4} on B, as well as the $\mathbb{Z}_2 \times \mathbb{Z}_2$ generating automorphisms τ_{B1} and τ_{B2}, are defined and discussed. In [20] and section 3, it is shown that generic $B = d\mathbb{P}_9$ surfaces allowing an involution α_B are elliptically fibered with projection map $\beta : B \to \mathbb{P}^1$ and are four-fold covers of a surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$. However, the strong restrictions on B necessary to allow a $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group, also produce extra structure on these surfaces. In section 4, it is shown that B can, alternatively, be expressed as a two-fold cover of another surface $\bar{Q} = \mathbb{P}^1 \times \mathbb{P}^1$. Using this result, we demonstrate in section 5 that the restricted surfaces are "ruled". That is, these surfaces have a second fibration with projection $\delta : B \to \mathbb{P}^1$ whose generic fiber is not elliptic but, rather, the projective space $\mathbb{P}^1$. These, admittingly technical, results play an important role. They allow one to identify, and give the properties of, a canonical set of generators of $H_2(B, \mathbb{Z})$, which we denote by e_i, $i = 1, \ldots, 9$ and l. This is done in section 6. Sections 7, 8 and 9 are devoted to computing the explicit transformation laws of these generators under the involutions $(-1)_B$, α_B, and t_{e_6}, t_{e_4} respectively. These results are then composed to produce the action of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group on the canonical set of generators. This is carried out in section 10, where an explicit table of the transformations of the generators $e_1, \ldots, e_9$ and l of $H_2(B, \mathbb{Z})$ under all relevant involutions is presented. Having found the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on $H_2(B, \mathbb{Z})$, we can search for the curve classes that are invariant under this action. In section 10, we show that these are generated by four invariant classes, which are presented explicitly. Finally, in section 11, using the construction outlined in section 2, we "lift" the four invariant classes on each of B and B' to the fiber product manifold X. By construction, these classes are in $H_4(X, \mathbb{Z})$. It is shown that $H_4(X, \mathbb{Z})$ has seven independent homology generators which are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group on X. These classes are, of course, constructed from the lift of invariant classes on B and B'. We explicitly exhibit these seven generators. Clearly, they survive the modding out of X by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. Therefore, the torus-fibered Calabi-Yau threefolds $Z = X/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ with fundamental group $\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_2$ admit a rank seven homology group $H_4(Z, \mathbb{Z})$, as required to produce anomaly free, three family standard-like models.

Having established these requisite results, we can turn to the problem of constructing stable, holomorphic vector bundles with structure group $G = \mathrm{SU}(4)$ on the Calabi-Yau threefolds Z. This will be accomplished in [21]. A detailed discussion of the associated particle physics will appear in [22, 23]. A more precise mathematical discussion of the entire program is also in preparation [24]. We emphasize to the reader that, although much of our discussion was within the context of the strongly coupled heterotic superstring, that is, M-theory, our results are equally applicable to weakly coupled heterotic vacua.

2. The Calabi-Yau threefolds Z

To make the paper self contained and to fix notation, we recall in this section the construction of torus fibered Calabi-Yau threefolds Z with fundamental group

$$\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_2 . \tag{2.1}$$

For a more complete exposition, the reader is referred to [20]. To construct torus fibered Calabi-Yau spaces Z with non-trivial first homotopy group, one starts with simply connected elliptically fibered Calabi-Yau threefolds X. Denote the projection map by

$$\pi : X \to B^{'} . \tag{2.2}$$

The generic fiber of π is isomorphic to a torus T^2. Furthermore, there exists a zero section $\sigma : B^{'} \to X$ which fixes a unique point on each fiber. This point acts as the identity of an abelian group, turning T^2 into an elliptic curve. In our construction, the base $B^{'}$ is chosen to be isomorphic to a rational elliptic surface $d\mathbb{P}_9$. Being a rational elliptic surface, $B^{'}$ is itself elliptically fibered over $\mathbb{P}^1$. We denote its projection map by

$$\beta^{'} : B^{'} \to \mathbb{P}^{1'} . \tag{2.3}$$

It can be shown, see for example [20], that every elliptically fibered Calabi-Yau threefold X over a base $B^{'}$ which is isomorphic to $d\mathbb{P}_9$ is actually a fiber product of two rational elliptic surfaces B and $B^{'}$. That is,

$$X = B \times_{\mathbb{P}^1} B^{'} , \tag{2.4}$$

where

$$B \times_{\mathbb{P}^1} B^{'} = \{(b, b^{'}) \in B \times B^{'} | \beta(b) = \beta^{'}(b^{'})\} \tag{2.5}$$

and $\beta : B \to \mathbb{P}^1$ and $\beta^{'} : B^{'} \to \mathbb{P}^{1'}$ are the projection maps of the two rational elliptic surfaces. Note that hidden in the definition of $B \times_{\mathbb{P}^1} B^{'}$ is an isomorphism $i : \mathbb{P}^1 \to \mathbb{P}^{1'}$ which identifies these spaces. The fiber product structure allows us to construct involutions τ_X on X in terms of involutions τ_B and $\tau_{B'}$ on B and $B^{'}$ respectively.

As proven in [20], when B and $B^{'}$ are each restricted to a two-parameter subset of $d\mathbb{P}_9$ surfaces, there exists a freely acting automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on X. This group is generated by two commuting involutions τ_{X1} and τ_{X1} defined as

$$\tau_{X1} = \tau_{B1} \times_{\mathbb{P}^1} \tau_{B'1} , \qquad \tau_{X2} = \tau_{B2} \times_{\mathbb{P}^1} \tau_{B'2} , \tag{2.6}$$

where τ_{Bi} and $\tau_{B'i}$ for $i = 1, 2$ are involutions on B and $B^{'}$. It follows that one can define a smooth quotient manifold

$$Z = \frac{X}{(\mathbb{Z}_2 \times \mathbb{Z}_2)} . \tag{2.7}$$

It was proven in [20, 24] that Z is indeed a Calabi-Yau threefold, albeit one with no global sections. Hence, it is only a torus fibration and is not elliptically fibered. Clearly, to understand these Calabi-Yau threefolds Z, it is essential to understand the properties of the Calabi-Yau manifolds X and the action of the automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$. However, it follows from the above discussion that the properties of X and its automorphism group are determined by the choice of the rational elliptic surfaces B and $B^{'}$, the involutions τ_{Bi} and $\tau_{B'i}$ for $i = 1, 2$ on these surfaces and the specification of an identification map $i : \mathbb{P}^1 \to \mathbb{P}^{1'}$. We turn, therefore, to the properties of rational elliptic surfaces and the classification of their involutions.

3. Rational elliptic surfaces

We begin this section by considering generic rational elliptic surfaces. As already mentioned, a rational elliptic surface B is an elliptic fibration over $\mathbb{P}^1$. We denote the projection map by

$$\beta : B \to \mathbb{P}^1 . \tag{3.1}$$

The generic fiber of β is isomorphic to a torus T^2. Furthermore, there is a zero section

$$e : \mathbb{P}^1 \to B \tag{3.2}$$

which fixes a unique zero point on each fiber, turning it into an elliptic curve. In addition, there exists a second projection map, denoted by β_2 in [20], such that

$$\beta_2 : B \to \mathbb{P}^2. \tag{3.3}$$

The pre-image of β_2 for a generic point $p \in \mathbb{P}^2$, that is $\beta_2^{-1}(p)$, consists of one point. However, at nine points, which we denote by ω_a for $a = 1, \dots, 9$,

$$\beta_2^{-1}(\omega_a) \equiv e_a \cong \mathbb{P}^1 . \tag{3.4}$$

Hence, B is a blow-up of $\mathbb{P}^2$ at nine points. This description of B is very useful in determining several geometric properties of the surface. We find that the generators for the homology classes of $H_2(B, \mathbb{Z})$ are the nine blow-up $\mathbb{P}^1$ lines $e_1, \dots, e_9$, also called exceptional divisors, and the pull-back $\beta_2^{-1}(l)$ of a line l in $\mathbb{P}^2$. We simply denote $\beta_2^{-1}(l)$ by l as well. That is, the lattice $H_2(B, \mathbb{Z})$ is spanned by these ten elements. Since B is connected, it follows that $H_0(B, \mathbb{Z}) \cong \mathbb{Z}$ and, hence, $H_0(B, \mathbb{Z})$ is generated by a point. Additionally, since B is compact and orientable, $H_4(B, \mathbb{Z}) \cong \mathbb{Z}$. Using, for example, the Van Kampen theorem, one can show that $H_1(B, \mathbb{Z})$ and $H_3(B, \mathbb{Z})$ vanish. Therefore, the topological Euler characteristic of B is given by

$$\chi(B) = \sum_{i=0}^{4} (-1)^i \mathrm{rank} H_i(B, \mathbb{Z}) = 1 + 10 + 1 = 12 . \tag{3.5}$$

It is easy to calculate the canonical bundle $K_B = \wedge^2(T^*B)$, where we have denoted the holomorphic cotangent bundle of B by T^*B. The result is

$$K_B = \beta_2^* K_{\mathbb{P}^2} \otimes \mathcal{O}_B \left(\sum_{i=1}^{9} e_i \right) = \mathcal{O}_B \left(-3l + \sum_{i=1}^{9} e_i \right) . \tag{3.6}$$

JHEP01(2004)059

As shown in [20], these geometric properties, derived using the projection $\beta_2 : B \to \mathbb{P}^2$, have consequences for the fibration $\beta : B \to \mathbb{P}^1$. The first consequence is that one can consider the exceptional divisors $e_1, \ldots, e_9$ as sections of this fibration. For concreteness, we choose the zero section to be

$$e = e_9 \,. \tag{3.7}$$

It was shown in [20] that the class of a generic fiber of (3.1) is given by

$$f = 3l - \sum_{i=1}^{9} e_i \,. \tag{3.8}$$

Therefore, (3.6) can be written as

$$K_B = \mathcal{O}_B(-f) \,. \tag{3.9}$$

Since the Euler characteristic of a smooth torus vanishes, the fact that $\chi(B) = 12$ requires the occurrence of singular fibers of (3.1). For generic surfaces B, these singular fibers are of Kodaira type [26] I_1 with $\chi(I_1) = 1$. Hence, there are twelve such fibers.

Having discussed the basic properties of generic rational elliptic surfaces B, we can proceed to determine all possible involutions on them. As shown in [20], all involutions

$$\tau_B : B \to B \tag{3.10}$$

induce an involution on the base $\mathbb{P}^1$, namely, $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$, obeying

$$\beta \circ \tau_B = \tau_{\mathbb{P}^1} \circ \beta \,. \tag{3.11}$$

Hence, τ_B maps each fiber f to some possibly different fiber $\tau_B(f)$. Since involutions $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ are either the identity map $\tau_{\mathbb{P}^1} = id$ or a non-trivial map with two fixed points, where the fixed points determine $\tau_{\mathbb{P}^1}$ uniquely, the induced action on the base $\mathbb{P}^1$ gives a crude classification of involutions on B. Let us first consider those involutions which act as the identity on the base.

One such involution is canonically given for any elliptic fibration, namely

$$(-1)_B : B \to B \,. \tag{3.12}$$

It is defined as follows. Since the zero section e fixes a unique zero on each fiber, which we denote by e as well, each fiber forms an abelian group. We will denote the operation of addition on each fiber by $\dot{+}$ and the operation of inverse on any element a by $\dot{-}a$. The restriction of $(-1)_B$ to a fiber maps each point a of that fiber to its abelian group inverse. Therefore, we obtain

$$(-1)_B|_f(a) = \dot{-}a \,. \tag{3.13}$$

Since the choice of the identity element e is determined by the global section e, $(-1)_B$ is defined globally. Furthermore, $(-1)_B$ induces an automorphism on the global sections of B. That is

$$(-1)_B : \Gamma(B) \to \Gamma(B) \,, \tag{3.14}$$

where $\Gamma(B)$ denotes the set of global sections of the fibration $\beta : B \to \mathbb{P}^1$. More specifically, for any section $\xi \in \Gamma(B)$

$$(-1)_B(\xi) = \dot{-}\xi\,, \tag{3.15}$$

where $\dot{-}\xi$ is defined as follows. Consider any point $p \in \mathbb{P}^1$ and let $\xi(p) = a$. Then the section $\dot{-}\xi$ is defined by $(\dot{-}\xi)(p) = \dot{-}a$.

Another automorphism of B which acts trivially on the base $\mathbb{P}^1$ is given, for any section ξ, by the translation t_ξ. If we denote the point of intersection of ξ with a fiber f by ξ as well, then the action of t_ξ is given by

$$t_\xi|_f : a \to a\dot{+}\xi \tag{3.16}$$

for each point a on f. Again, the action t_ξ is globally defined and induces an automorphism on the global sections of B, namely

$$t_\xi(s) = s\dot{+}\xi \tag{3.17}$$

for any section s in $\Gamma(B)$. For $p \in \mathbb{P}^1$ and $s(p) = a$, $s\dot{+}\xi$ is defined by $(s\dot{+}\xi)(p) = a\dot{+}\xi$. Note that (3.15) and (3.17) induce an abelian group structure on $\Gamma(B)$. Since this structure is defined in terms of the group law on each fiber, we continue to denote addition and inverse on $\Gamma(B)$ by $\dot{+}$ and $\dot{-}$.[1] It is clear, that t_ξ is generically not an involution. A necessary condition for it to be an involution is that the section ξ intersect each fiber at a point of order two, that is, $\xi\dot{+}\xi = e$. There are four such points on each fiber, which we denote by $e, e^{'}, e^{''}$ and $e^{'''}$. Of course, the identity element e is one such point. Observe that points of order two are also fixed under the action of $(-1)_B$. For example

$$(-1)_B(e^{'}) = \dot{-}e^{'} = (e^{'}\dot{+}e^{'})\dot{-}e^{'} = e^{'}\,. \tag{3.18}$$

However, the above condition is not sufficient to guarantee that global sections ξ with $t_\xi^2 = id$ exist. In fact, for generic elliptic surfaces B they do not. They can occur, however, in a restricted subset of surfaces B. We will use such restricted surfaces to construct the Calabi-Yau threefolds X.

A further type of involution τ_B on B which induces a trivial action on the base $\mathbb{P}^1$ is given by

$$\tau_B = t_\xi \circ (-1)_B\,. \tag{3.19}$$

However, since it is impossible to lift these involutions to a fixed point free involution on X, we will not consider them further.

We now discuss involutions on B which act non-trivially on the base $\mathbb{P}^1$. By a theorem of [12], each such involution can be written as

$$\tau_B = t_\xi \circ \alpha_B\,, \tag{3.20}$$

[1] It is helpful at this point to discuss our notation in more detail. On any fibered surface B, a section $\xi \in \Gamma(B)$ will determine a homology class, also denoted by ξ, in $H_2(B, \mathbb{Z})$. There is a natural notion of addition and inverse in $H_2(B, \mathbb{Z})$ which we denote everywhere in this paper by $+$ and $-$. For generic fibered surfaces, however, there is no group law on $\Gamma(B)$. When B is elliptically fibered, a notion of addition and inverse on $\Gamma(B)$ does exist. To distinguish this from the group law on $H_2(B, \mathbb{Z})$, we denote the group law on $\Gamma(B)$ by $\dot{+}$ and $\dot{-}$.

JHEP01(2004)059

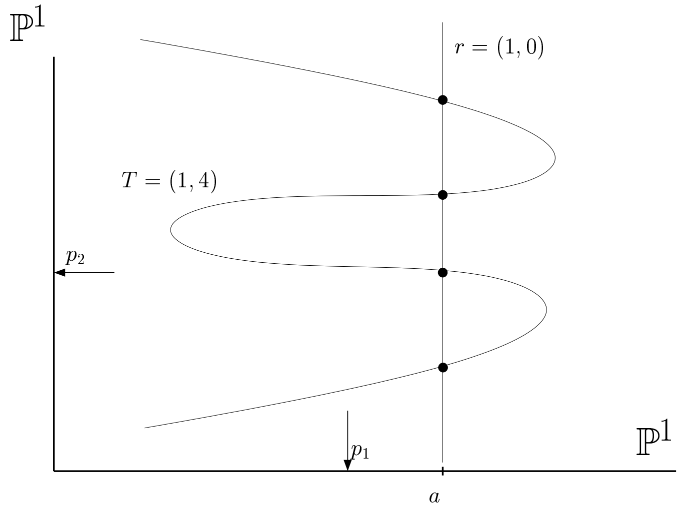

Figure 1: The bidegree $(2,4)$ curve $M = T \cup r$ in $Q = \mathbb{P}^1 \times \mathbb{P}^1$. The projections $p_i : Q \to \mathbb{P}^1_i, i = 1,2$ are explicitly shown.

where $\alpha_B : B \to B$ is an involution of B leaving the zero section e fixed and inducing a non-trivial action on $\mathbb{P}^1$. That is, $\tau_{\mathbb{P}^1} \circ \beta = \beta \circ \alpha_B$ where $\tau_{\mathbb{P}^1}$ is a non-trivial involution on $\mathbb{P}^1$ with two fixed points, which we call 0 and ∞. The section ξ must have the property that

$$\alpha_B(\xi) = (-1)_B(\xi)\,. \tag{3.21}$$

As shown in [20], a generic rational elliptic surface B does not admit such involutions α_B. However, there is five dimensional sub-family in the eight dimensional family of rational elliptic surfaces which does allow such involutions. Furthermore, it was shown in [12] that each member of this five dimensional sub-family has a rank four lattice of sections ξ fulfilling condition (3.21).

As described in [20], α_B leaves two fibers of B stable, namely $f_0 = \beta^{-1}(0)$ and $f_\infty = \beta^{-1}(\infty)$. Furthermore, it was shown that α_B acts as the identity map on f_0 and as $(-1)_B$ on f_∞. Hence, the double cover

$$\kappa : B \to \frac{B}{\alpha_B} \tag{3.22}$$

has the image of f_0 and the image of the four points of order two of the fiber f_∞ under κ as its branch locus. Hence, the image of f_∞, that is $f_\infty/\alpha_B \subset B/\alpha_B$, contains four isolated A_1 surface singularities of B/α_B. These can be resolved by blowing up and one obtains an I_0^* fiber. It was shown in [20] that, for generic B in the five parameter family of rational elliptic surfaces, the quotient B/α_B can be described as the double cover of $Q = \mathbb{P}^1 \times \mathbb{P}^1$. That is

$$\pi : \frac{B}{\alpha_B} \to Q\,, \tag{3.23}$$

where the branch locus M of π consists of the union of a $(1,4)$ curve T and a $(1,0)$ curve r. These are shown in figure 1, where the two natural projections of Q, $p_i : Q \to \mathbb{P}^1_i$, $i = 1,2$ to both $\mathbb{P}^1$ factors of Q are also defined.

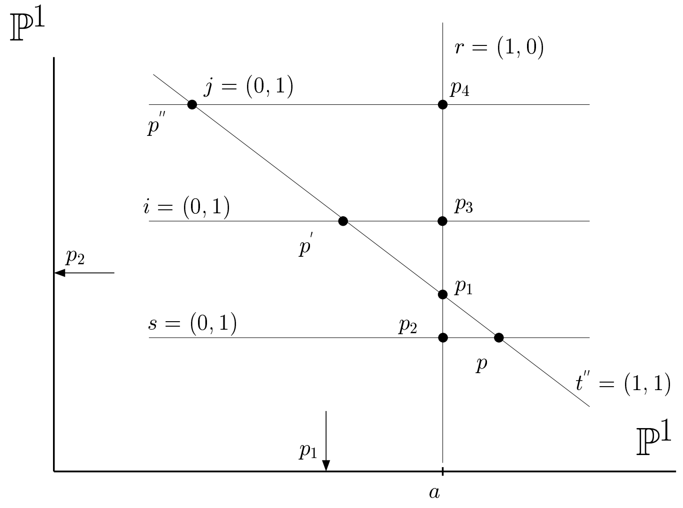

Figure 2: A schematic representation of curve $M = t'' \cup s \cup i \cup j \cup r$ in $Q = \mathbb{P}^1 \times \mathbb{P}^1$.

Having described all possible involutions on rational elliptic surfaces B, we now further restrict ourselves to a two dimensional sub-family of rational elliptic surfaces. These were used in [20] to construct Calabi-Yau threefolds X with two freely acting involutions. Since this two dimensional family is a subset of the five dimensional family mentioned above, these surfaces continue to admit involutions α_B. However, for these restricted surfaces, the curve T splits into $T = t'' \cup s \cup i \cup j$, where t'' is a $(1,1)$ curve and s, i and j are $(0,1)$ rulings. See figure 2 for a pictorial description of them. Note that t'' intersects each of r, s, i and j at one point. Call these points p_1, p, p' and p'' respectively. Furthermore, s, i and j each intersect r at a single point, which we denote by as p_2, p_3 and p_4.

What is the double cover of Q corresponding to this restricted branch locus? For maximal clarity, we will here denote this double cover by W_M, relating it to B/α_B only later. We continue to indicate the covering map by $\pi : W_M \to Q$. Let us consider the composition map $\tilde{p}_1 = p_1 \circ \pi$. Clearly, we have the projection map

$$\tilde{p}_1 : W_M \to \mathbb{P}^1 \,. \tag{3.24}$$

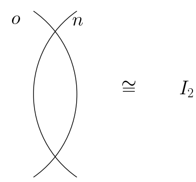

Figure 3: The reducible fiber I_2, where o denotes the proper transform and n the exceptional divisor.

The fiber of $\tilde{p}_1$, for a generic point $x \in \mathbb{P}^1$, is the double cover of the fiber of projection p_1 at this point, branched over the four points where this fiber intersects the branch locus M, namely, at its intersection with t'', s, j and i. It was shown in [20] that this generic fiber of $\tilde{p}_1$ is isomorphic to a torus. Hence, W_M is a torus fibration. What is the double cover of the fiber corresponding to the point $p_1(r) = a$? Since r is in the branch locus itself, $\tilde{p}_1^{-1}(a)$ is a double $\mathbb{P}^1$ line. Furthermore, since r intersects the other components of the branch curve M, namely, t'', s, i and j at p_1, p_2, p_3 and p_4 respectively, W_M has four A_1 surface singularities on $\tilde{p}_1^{-1}(a)$ located at $\pi^{-1}(p_i)$ for $i = 1, \ldots, 4$. Recall from [20] that resolving these singularities would produce an I_0^* fiber. In addition to $\tilde{p}_1^{-1}(a)$, there are three other fibers of $\tilde{p}_1$ which contain singular points of W_M. Singular points occur when different

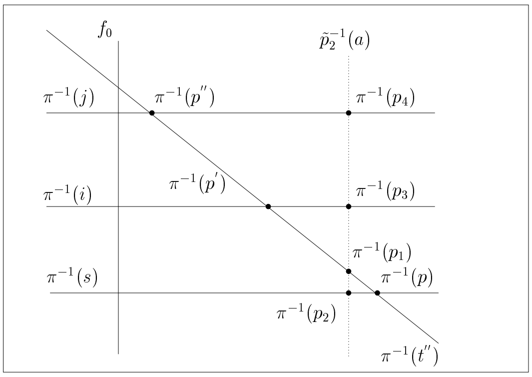

Figure 4: The elliptic fibration W_M with the four distinguished sections.

components of the branch locus of W_M intersection Clearly, the remaining intersection points are in the fibers $p_1^{-1}(p_1(p))$, $p_1^{-1}(p_1(p'))$ and $p_1^{-1}(p_1(p''))$. As explained in [20], the double covers of these three fibers in W_M are singular, each containing an A_1 surface singularity. After resolving the singularities, each of these fibers consists of two components, the proper transform of the original fiber and an exceptional divisor which is isomorphic to $\mathbb{P}^1$. This is called an I_2 fiber in the Kodaira classification and is sketched in figure 3. To conclude, the fibration (3.24) has four singular fibers, one containing four A_1 singularities and the remaining three with one A_1 singularity each. Resolving these singularities, we obtain a smooth surface, which we denote by $\widehat{W_M}$, with one I_0^* fiber and three I_2 fibers. Recall from [20] that $\chi(I_0^*) = 6$ and $\chi(I_2) = 2$. Therefore $\chi(\widehat{W_M}) = 12$. Furthermore, the fibration $\tilde{p}_1 : W_M \to \mathbb{P}^1$ has four special sections, namely $\pi^{-1}(s), \pi^{-1}(i)$, $\pi^{-1}(j)$ and $\pi^{-1}(t'')$. These sections, each of which is isomorphic to $\mathbb{P}^1$, are described graphically in figure 4. Choosing one of them, which we take to be $\pi^{-1}(s)$, as the zero section completes the proof that $\widehat{W_M}$ is a rational elliptically fibered surface with one I_0^* fiber and three I_2. As discussed in [20], W_M is its Weierstrass model. We can summarize the projections π, p_1 and $\tilde{p}_1$ in the diagram

$$\begin{tikzcd} W_M \arrow[r, "\pi"] \arrow[d, "\tilde{p}_1"'] & Q \arrow[d, "p_1"] \\ \mathbb{P}^1 \arrow[r, "\mathrm{id}"'] & \mathbb{P}^1 \end{tikzcd} \tag{3.25}$$

Recall from [20] that one can choose a smooth fiber, which we call f_0, in W_M and construct the double cover of W_M branched over f_0 and the four singular points in $\tilde{p}_1^{-1}(a)$. One then obtains the double cover of W_M, which we denote by $\bar{B}$ with the cover map

$$\bar{\kappa} : \bar{B} \to W_M \, . \tag{3.26}$$

This double covering map $\bar{\kappa}$ induces an involution $\alpha_{\bar{B}}$ on $\bar{B}$, which acts non-trivially as $\tau_{\mathbb{P}^1}$ on the base $\mathbb{P}^1$. Defining

$$sq : \mathbb{P}^1 \to \frac{\mathbb{P}^1}{\tau_{\mathbb{P}^1}} \tag{3.27}$$

JHEP01(2004)059

and noting that $\mathbb{P}^1/\tau_{\mathbb{P}^1} \cong \mathbb{P}^1$, we obtain

$$\begin{tikzcd} \bar{B} \arrow[r, "\bar{\kappa}"] \arrow[d, "\bar{\beta}"'] & W_M \arrow[d, "\tilde{p}_1"] \\ \mathbb{P}^1 \arrow[r, "\mathrm{sq}"'] & \mathbb{P}^1 \end{tikzcd} \quad , \tag{3.28}$$

which defines the fibration $\bar{\beta} : \bar{B} \to \mathbb{P}^1$. It is important to note that $\bar{B}$, the double cover of W_M, is not yet the surface B. The reason is the following.

Let us recall from [20] some of the properties of $\bar{B}$. First, from the diagram (3.28) it is easy to see that a generic fiber of W_M has two disjoint pre-images in $\bar{B}$. These form two fibers of the fibration $\bar{\beta} : \bar{B} \to \mathbb{P}^1$ and get exchanged by the involution $\alpha_{\bar{B}}$. Hence, only the fibers $\bar{\kappa}^{-1}(f_0)$ and $\bar{\kappa}^{-1}(\tilde{p}_1^{-1}(a))$ are stable in $\bar{B}$. We denote these fibers by f_0 and f_∞ respectively. Of course, these are the fibers over the fixed points 0 and ∞ of $\tau_{\mathbb{P}^1}$, the involution on $\mathbb{P}^1$ induced by $\alpha_{\bar{B}}$. Although $\tilde{p}_1^{-1}(a)$ in W_M contains four surface singularities of type A_1, its pre-image f_∞ in $\bar{B}$ is smooth. This is explained in [20] and follows from the fact that the pre-images of the four singularities of $\tilde{p}_1^{-1}(a)$ in W_M are fixed under the covering involution $\alpha_{\bar{B}}$. Be that as it may, $\bar{B}$ does have singular fibers. Recall that W_M had three additional singular fibers containing an A_1 surface singularity. Since these fibers are generically not f_0 and $\tilde{p}_1^{-1}(a)$ in W_M, their pre-images each consists of two fibers in $\bar{B}$. Clearly, each such fiber contains an A_1 surface singularity. Therefore, there are six fibers in $\bar{B}$ containing a surface singularity of type A_1. We conclude that the surface $\bar{B}$ is singular, whereas the rational elliptic surface B is smooth. Therefore, B differs from $\bar{B}$, being obtained from it by blowing up each of the six A_1 singularities, as we will discuss below.

What are the pre-images of the four special sections of W_M? First consider $\pi^{-1}(s)$. Recall that $\pi^{-1}(s)$ intersects the fiber f_0 in W_M and contains the singular point $\pi^{-1}(p_2)$. Hence, it intersects the branch locus of $\bar{B}$ in two points and its pre-image $\bar{\kappa}^{-1}(\pi^{-1}(s))$ is the double cover of $\pi^{-1}(s)$. Since $\pi^{-1}(s) \cong \mathbb{P}^1$, it follows that $\bar{\kappa}^{-1}(\pi^{-1}(s))$ is isomorphic to $\mathbb{P}^1$ as well. Furthermore, since $\bar{\kappa}^{-1}(\pi^{-1}(s))$ intersects each fiber of $\bar{B}$ once, it is a section of the fibration $\bar{\beta} : \bar{B} \to \mathbb{P}^1$. Note that $\bar{\kappa}^{-1}(\pi^{-1}(s))$ also contains two A_1 surface singularities, namely the pre-image $\bar{\kappa}^{-1}(\pi^{-1}(p))$. The pre-images under $\bar{\kappa}$ of $\pi^{-1}(i)$ and $\pi^{-1}(j)$ have similar properties. They form sections of $\bar{B}$ each containing two A_1 surface singularities, namely the two points $\bar{\kappa}^{-1}(\pi^{-1}(p'))$ and the two points $\bar{\kappa}^{-1}(\pi^{-1}(p''))$ respectively. The pre-image $\bar{\kappa}^{-1}(\pi^{-1}(t''))$ in $\bar{B}$, the double cover of $\pi^{-1}(t'')$ branched over $\pi^{-1}(p_1)$ and $f_0 \cap \pi^{-1}(t'')$, is isomorphic to $\mathbb{P}^1$ as well. Clearly, it is also a section of $\bar{B}$. However, unlike the other three sections $\bar{\kappa}^{-1}(\pi^{-1}(s))$, $\bar{\kappa}^{-1}(\pi^{-1}(i))$ and $\bar{\kappa}^{-1}(\pi^{-1}(j))$, $\bar{\kappa}^{-1}(\pi^{-1}(t''))$ contains all six A_1 surface singularities. This follows from the fact that $\pi^{-1}(t'')$ contains $\pi^{-1}(p)$, $\pi^{-1}(p')$ and $\pi^{-1}(p'')$.

Having discussed the singular surfaces $\bar{B}$, we finally can construct their resolution, the rational elliptic surfaces B that we wish to consider. If we denote by B the blow-up of $\bar{B}$

at the six A_1 singularities and let $c : B \to \bar{B}$ be the projection map, then clearly

$$\begin{tikzcd} B \arrow[r, "c"] \arrow[d, "\beta"'] & \bar{B} \arrow[d, "\bar{\beta}"] \\ \mathbb{P}^1 \arrow[r, "\mathrm{id}"'] & \mathbb{P}^1 \end{tikzcd} , \tag{3.29}$$

which defines the fibration $\beta : B \to \mathbb{P}^1$ as $\beta = \bar{\beta} \circ c$. Combining diagrams (3.28) and (3.29), we see that

$$\kappa : B \to W_M \tag{3.30}$$

is the covering of W_M by B where $\kappa = \bar{\kappa} \circ c$. We denote by α_B the induced involution on B of $\alpha_{\bar{B}}$.

What are the fibers of B? As described in [20], each singular fiber with an A_1 surface singularity can be resolved into an I_2 fiber. Hence, B has six I_2 fibers, namely, the pre-images of $\kappa^{-1}(\tilde{p}_1^{-1}(p_1(p)))$, $\kappa^{-1}(\tilde{p}_1^{-1}(p_1(p^{'})))$ and $\kappa^{-1}(\tilde{p}_1^{-1}(p_1(p^{''})))$. Let us denote the exceptional divisors corresponding to $\kappa^{-1}(\tilde{p}_1^{-1}(p_1(p)))$ by n_1 and n_2, the exceptional divisors corresponding to $\kappa^{-1}(\tilde{p}_1^{-1}(p_1(p^{'})))$ by n_3 and n_4 and the exceptional divisors corresponding to $\kappa^{-1}(\tilde{p}_1^{-1}(p_1(p^{''})))$ by n_5 and n_6. The associated proper transforms are denoted $o_1, \ldots, o_6$. Hence, the six I_2 fibers in B have the structure $n_i \cup o_i$, $i = 1, \ldots, 6$.

Next, consider the proper transform of the four distinguished sections in $\bar{B}$. Clearly, they are still sections of B. Following the notation of [20], we denote the proper transform of $\kappa^{-1}(\pi^{-1}(s))$ by e_9, the proper transform of $\kappa^{-1}(\pi^{-1}(i))$ by e_6 and the proper transform of $\kappa^{-1}(\pi^{-1}(j))$ by e_4. From the distribution of the singularities at the image of these sections in $\bar{B}$, as described above, it is clear that e_9 intersects n_1 and n_2, e_6 intersects n_3 and n_4 and e_4 intersects n_5 and n_6. We will show in the next section that the proper transform of $\kappa^{-1}(\pi^{-1}(t^{''}))$ is $e_4 \dot{+} e_6$.

At this point, we want to make an important technical comment. As discussed above, $\bar{B}$ is the singular four-fold cover of Q. B, on the other hand, is a smooth surface obtained by blowing up the singularities in $\bar{B}$. The exact relationship is expressed in (3.29). However, in most of the remaining discussion in this paper, one can readily distinguish whether a surface is $\bar{B}$ or its blow-up B from context. With this in mind, we find that it leads to enormous simplification of our notation if we simply use the symbol B to denote both B and $\bar{B}$. In keeping with this notation, we will also denote both sets of mappings $\bar{\beta}, \bar{\kappa}, \alpha_{\bar{B}}$ and β, κ, α_B by β, κ, α_B. We do this henceforth. With this simplified notation, we can identify

$$W_M \cong \frac{B}{\alpha_B} , \tag{3.31}$$

as in (3.22) and (3.23). In a few places when it is helpful to specify the surface and the associated mappings exactly, we will do so.

As in [20], we choose e_9 to be the zero section of the rational elliptic surface B. Furthermore, as we will show in the next section, e_4 and e_6 are sections of B which intersect each fiber at a point of order two. Hence

$$(-1)_B(e_9) = e_9 , \qquad (-1)_B(e_6) = e_6 , \qquad (-1)_B(e_4) = e_4 . \tag{3.32}$$

It is clear from the construction that e_9, e_6 and e_4 are invariant under α_B. Hence, they fulfill condition (3.21). Note, in addition, that $e_4\dot{+}e_6$ is a section of B which intersects each fiber at a point of order two. It follows that it too satisfies (3.21). Using these sections, we can define the two involutions

$$\tau_{B1} = t_{e_6} \circ \alpha_B \,, \qquad \tau_{B2} = t_{e_4} \circ \alpha_B \,. \tag{3.33}$$

Using the fact that

$$\tau_{B1} \circ \tau_{B2} = t_{e_4 \dot{+} e_6} \tag{3.34}$$

and

$$t^2_{e_4 \dot{+} e_6} = id \,, \tag{3.35}$$

it was shown in [20] that τ_{B1} and τ_{B2} commute. It was explained in [20] that one can lift these commuting involutions to freely acting involutions on the Calabi-Yau threefolds X. Henceforth, we will assume (3.33) to be the generators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group on the rational elliptic surface B.

We choose the rational elliptic surface B' to be within the same two dimensional family. We denote the components of its I_2 fibers by n_i' and o_i', $i = 1, \ldots, 6$. The zero section is denoted by $e' \equiv e_9'$ and the other sections of order two by e_6', e_4' and $e_4'\dot{+}e_6'$. The generators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group on B' are then given by

$$\tau_{B'1} = t_{e_6'} \circ \alpha_B' \,, \qquad \tau_{B'2} = t_{e_4'} \circ \alpha_B' \,. \tag{3.36}$$

This concludes our review of rational elliptic surfaces. As discussed in section 2, the fiber product $X = B \times_{\mathbb{P}^1} B'$ is a Calabi-Yau threefold and admits the two commuting involutions

$$\tau_{X1} = \tau_{B1} \times_{\mathbb{P}^1} \tau_{B'1} \,, \qquad \tau_{X2} = \tau_{B2} \times_{\mathbb{P}^1} \tau_{B'2} \,. \tag{3.37}$$

They can be shown [20] to generate a freely acting automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on X. Hence, the quotient manifold

$$Z = \frac{X}{(\mathbb{Z}_2 \times \mathbb{Z}_2)} \tag{3.38}$$

is smooth and, as shown in [20, 24], is a Calabi-Yau threefold.

4. B as a double cover of $\bar{Q} = \mathbb{P}^1 \times \mathbb{P}^1$

In this section, we begin the process of computing the transformation laws of the elements of $H_4(X, \mathbb{Z})$ of the Calabi-Yau threefolds X under the automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$. To do this, we need to find the transformation laws of the second homology groups of the rational elliptic surfaces B and B' under the respective $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphisms.

Let us consider a rational elliptic surface within the two dimensional sub-family described in the previous section. To find the transformation laws of $H_2(B, \mathbb{Z})$ under $\mathbb{Z}_2 \times \mathbb{Z}_2$, we must first identify a set of generating elements of $H_2(B, \mathbb{Z})$. For a generic surface, B is a blow-up of $\mathbb{P}^2$ at nine separated points. Hence, for a generic B one can choose the nine exceptional divisors $e_1, \ldots, e_9$ and l, the pre-image of a line in $\mathbb{P}^2$, as a set of generators for $H_2(B,\mathbb{Z})$. Note that each of these exceptional curve is isomorphic to $\mathbb{P}^1$ and, hence, irreducible. Furthermore, the intersection numbers of $e_1,\ldots,e_9$ and l are given by

$$e_i \cdot e_j = -\delta_{ij}\,, \qquad e_j \cdot l = 0\,, \qquad l \cdot l = 1\,, \qquad i,j = 1,\ldots,9\,. \tag{4.1}$$

However, the rational elliptic surfaces we are going to consider are not generic. Rather, they are chosen to be in a restricted two dimensional sub-family. It can be shown that one effect of this restriction is to collide together several of the nine points in $\mathbb{P}^2$. That is, not all of the nine points are distinct. The blow-ups of these degenerate points continue to contain exceptional curves e_i, but their properties and linear dependence are more complicated. It turns out that for surfaces B in the two dimensional sub-family, l and the nine curves $e_1,\ldots e_9$ with intersection numbers (4.1) still form a set of generators of $H_2(B,\mathbb{Z})$. However, these curves are no longer necessarily irreducible, that is, they may consists of several components. This will be discussed in detail in section 6. This fact greatly complicates the process of determining their transformation laws under the $\mathbb{Z}_2\times\mathbb{Z}_2$ automorphisms. So much so, that it becomes necessary to use properties of rational elliptic surfaces not discussed in [20] or in the previous section.

To find the transformation laws of $e_1,\ldots e_9$ under $\mathbb{Z}_2\times\mathbb{Z}_2$, it is important to notice that, in addition to the fibration $\beta: B\to\mathbb{P}^1$ discussed in [20] and section 3, B admits a second fibration

$$\delta : B \to \mathbb{P}^1\,, \tag{4.2}$$

where the generic fiber of δ is isomorphic to $\mathbb{P}^1$ and not to a torus. This second fibration is very helpful in determining the $\mathbb{Z}_2\times\mathbb{Z}_2$ transformation laws since, first of all, its fibers turn out all to be invariant under the involution α_B and, secondly, all curves $e_1,\ldots e_9$ with exception of e_7 are contained in some of its singular fibers. This fibration is described in great detail in section 5. However, as a prerequisite, we need to prove the following property of B. Recall from section 3 that B is a four-fold cover, with mapping $\pi\circ\kappa$, of $Q=\mathbb{P}^1\times\mathbb{P}^1$. In this section, we will show that, viewed differently, B can actually be expressed as a double cover of another $\mathbb{P}^1\times\mathbb{P}^1$ surface, which we denote by $\bar{Q}$. This result will then allow us to construct the fibration $\delta: B\to\mathbb{P}^1$.

We begin by analyzing the covering map $\pi: W_M\to Q$ more closely. Recall from section 3 that B is a double cover of W_M with covering map $\kappa: B\to W_M$. By construction, the involution on B which is associated with this covering is α_B. That is, $W_M\cong B/\alpha_B$. One refers to α_B as the covering involution of the map κ. Next we need to analyze the covering involution on W_M associated with the map $\pi: W_M\to Q$? Determining this will afford us deeper insight into the structure of B. To find this involution, first remember that

$$\begin{array}{ccc} W_M & \xrightarrow{\pi} & Q \\ {\scriptstyle \tilde{p}_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ \mathbb{P}^1 & \xrightarrow[\mathrm{id}]{} & \mathbb{P}^1 \end{array}\,, \tag{4.3}$$

where the branch locus of π in Q is $M = s\cup i\cup j\cup t''\cup r$. In addition, recall that the branch locus contains the image of the zero section, $\pi^{-1}(s)$, of W_M. It is clear from this

$$
\begin{tikzcd}
 & B \arrow[dl, "\kappa"'] \arrow[d, "\psi"'] \arrow[dr, "\kappa'"] & \\
B/\alpha_B \cong W_M \arrow[dr, "\pi"'] & B/(-1)_B \cong \bar{Q} \arrow[d, "\Delta"'] & W_{M'} \cong B/((-1)_B \circ \alpha_B) \arrow[dl, "\pi'"] \\
 & Q \cong W_M/(-1)_{W_M} \cong \bar{Q}/\alpha_{\bar{Q}} \cong W_{M'}/(-1)_{W_{M'}} &
\end{tikzcd}
$$

Figure 5: Three different ways to consider B as a degree four cover of Q.

and the structure of diagram (4.3) that the covering involution on W_M of π must act as the identity on the base $\mathbb{P}^1$ and leave the zero section point-wise fixed. There is a unique involution on W_M with this property. To see this, consider the involution $(-1)_B$ on B defined in section 3. Clearly, this induces a unique involution $(-1)_{W_M}$ on W_M under the covering map $\kappa : B \to W_M$. It is straightforward to see that $(-1)_{W_M}$ has the required properties and, hence, is the covering involution for $\pi : W_M \to Q$. That is

$$Q \cong \frac{W_M}{(-1)_{W_M}} \, . \tag{4.4}$$

Combining this result with the discussion in section 3, we see that B is a four-fold cover of Q given by

$$B \xrightarrow{\kappa} W_M \cong \frac{B}{\alpha_B} \xrightarrow{\pi} Q \cong \frac{W_M}{(-1)_{W_M}} \, . \tag{4.5}$$

Note that we have first modded out by α_B and then by $(-1)_{W_M}$. This is shown graphically as the left hand routing in figure 5.

Expressing B as the four-fold covering of Q in this manner leads to the following important insight. Namely, note that it should not matter whether we first mod out by α_B and then by $(-1)_{W_M}$, the involution induced on W_M from $(-1)_B$, as in (4.5), or first mod out by $(-1)_B$ and then by the induced involution of α_B on $B/(-1)_B$. Either approach represents B as the same four-fold covering of Q. Let us specify this second decomposition of the covering more concretely. First, we denote

$$\frac{B}{(-1)_B} \cong \bar{Q} \tag{4.6}$$

JHEP01(2004)059

and the covering map associated with involution $(-1)_B$ by

$$\psi : B \to \bar{Q}\,. \tag{4.7}$$

Now denote the induced involution of α_B on $\bar{Q}$ by $\alpha_{\bar{Q}}$. Modding out $\bar{Q}$ by $\alpha_{\bar{Q}}$ must produce a surface isomorphic to Q. That is,

$$\frac{\bar{Q}}{\alpha_{\bar{Q}}} \cong Q\,. \tag{4.8}$$

We write the associate covering map as

$$\Delta : \bar{Q} \to Q\,. \tag{4.9}$$

Combining these results, we see that the four-fold covering of Q by B can be expressed in an alternative manner to (4.5) given by

$$B \xrightarrow{\psi} \bar{Q} \cong \frac{B}{(-1)_B} \xrightarrow{\Delta} Q \cong \frac{\bar{Q}}{\alpha_{\bar{Q}}}\,. \tag{4.10}$$

Here, we have first modded out by $(-1)_B$ and then by $\alpha_{\bar{Q}}$, which is induced from α_B on B. This is shown graphically as the middle routing in figure 5. Expressions (4.5) and (4.10) are not the only ways of writing B as the four-fold cover of Q. A third way is as follows. First mod out by the composite involution $(-1)_B \circ \alpha_B$ on B to produce the surface

$$\frac{B}{((-1)_B \circ \alpha_B)} \cong W_{M'} \tag{4.11}$$

and the covering map

$$\kappa^{'} : B \to W_{M'}\,. \tag{4.12}$$

Now denote the induced involution of $(-1)_B$ on $W_{M'}$ by $(-1)_{W_{M'}}$. Modding out $W_{M'}$ by $(-1)_{W_{M'}}$ must produce a surface isomorphic to Q. That is,

$$\frac{W_{M'}}{(-1)_{W_{M'}}} \cong Q\,. \tag{4.13}$$

We write the associated covering map as

$$\pi^{'} : W_{M'} \to Q\,. \tag{4.14}$$

Combining these results, we see that

$$B \xrightarrow{\kappa^{'}} W_{M'} \cong \frac{B}{((-1)_B \circ \alpha_B)} \xrightarrow{\pi^{'}} Q \cong \frac{W_{M'}}{(-1)_{W_{M'}}}\,, \tag{4.15}$$

where we have first modded out by $(-1)_B \circ \alpha_B$ and then by $(-1)_{W_{M'}}$, which is induced from $(-1)_B$ on B. This is shown graphically as the right hand routing in figure 5.

What are the branch loci of κ, π in (4.5), ψ, Δ in (4.10) and $\kappa^{'}$, $\pi^{'}$ in (4.15)? The branch loci of κ and π were discussed in the previous section and are pictured in the left

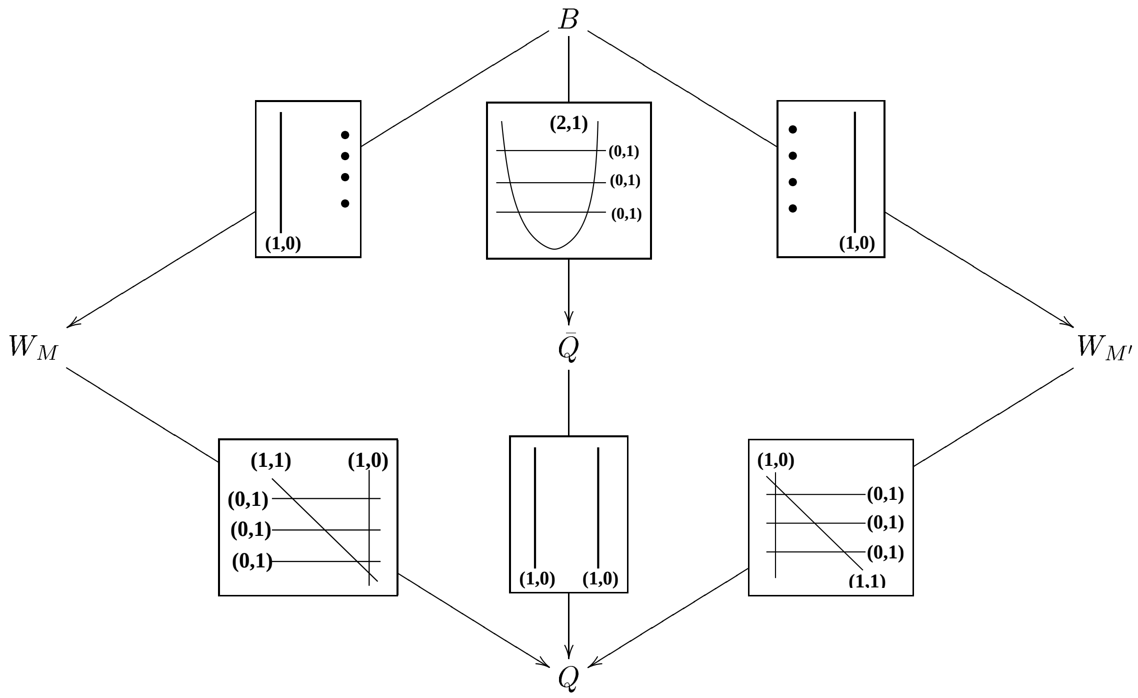

Figure 6: W_β as a double cover of a quadric.

hand routing of figure 6. Let us now discuss the branch loci for κ' and π', beginning with κ'. The covering involution for κ' is $(-1)_B \circ \alpha_B$. Since α_B acts as the identity map on f_0 and as $(-1)_B$ on f_∞, the composition $(-1)_B \circ \alpha_B$ acts as $(-1)_B$ on f_0 and as the identity map on f_∞. Hence, the branch locus for κ' in W'_M consists of the image of the four fixed points under $(-1)_B$ on f_0 and the fiber f_∞. Hence, W'_M has similar properties to W_M, one fiber f_0 which contains four A_1 surface singularities and which can be resolved into an I_0^* fiber, and three singular fibers which can be resolved into I_2 fibers. The branch locus in Q of the double cover π' now clearly consists of the three $(0,1)$ rulings s, i and j, the $(1,1)$ curve t'' and the image $\pi'(f_0)$ of the fiber f_0 in W'_M, which is a $(1,0)$ ruling that we denote by r'. That explains the right hand routing of figure 6. Finally, we consider the branch loci of ψ and Δ. Let us start with Δ. To proceed, we need to invoke a mathematical theorem which states that the branch locus of Δ in Q consists of the union of the branch loci of π and π' in Q minus their intersection. Hence, the branch locus of Δ in Q consists of the two $(1,0)$ rulings r and r'. Now consider the covering map ψ. To find its branch locus, we first must elucidate the structure of surface $\bar{Q}$ more explicitly. To do this, recall that $Q = \mathbb{P}^1 \times \mathbb{P}^1$ has two natural projections $p_i : Q \to \mathbb{P}^1_i$ for $i = 1, 2$ to the ith $\mathbb{P}^1$ component. Choose any point $x \in \mathbb{P}^1_2$ and consider the fiber $\Delta^{-1}(p_2^{-1}(x))$ in $\bar{Q}$. It is the double cover the fiber $p_2^{-1}(x)$, which is isomorphic to $\mathbb{P}^1$, branched over the two points where $p_2^{-1}(x)$ intersects the rulings r and r'. Hence $\Delta^{-1}(p_2^{-1}(x))$ is isomorphic to $\mathbb{P}^1$. It follows that $\bar{Q}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. That is

$$\bar{Q} = \mathbb{P}^1 \times \mathbb{P}^1 . \tag{4.16}$$

JHEP01(2004)059

We denote the two natural projections of $\bar{Q}$ by $\bar{p}_i : \bar{Q} \to \mathbb{P}^1_i$ for $i = 1, 2$. Of course, $\bar{Q}$ is smooth since the branch locus of $\bar{Q}$ in Q is smooth. With this information, we can now determine the branch locus of the double cover ψ. It is simply the pre-image under Δ of the curves s, i, j and t''. Since each of the rulings s, i and j pulls back to a unique fiber of $\bar{p}_2 : \bar{Q} \to \mathbb{P}^1$, they are $(0,1)$ rulings in $\bar{Q}$. The pre-image of t'' is isomorphic to $\mathbb{P}^1$ as well, since t'' intersects the branch locus for Δ in two points, namely $t'' \cap r$ and $t'' \cap r'$. However, t'' intersects the generic fiber of $p_2 : Q \to \mathbb{P}^1$ at a point not contained in the branch locus of Q. Therefore, its pre-image $\Delta^{-1}(t'')$ intersects the fiber $\bar{p}_2^{-1}(x)$ in two points. It follows that $\Delta^{-1}(t'')$ is a bidegree $(2,1)$ curve in $\bar{Q}$. The branch loci for Δ and ψ are described in the middle routing of figure 6. This finishes our explanation of the branch loci of B as a degree four cover of Q. We have shown that there are three different ways of describing it, a fact that will prove to be important later when, for a chosen set of generating elements of $H_2(B, \mathbb{Z})$, we try to compute their $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation laws.

We will finish this section with the proof that the sections e_4 and e_6 of $\beta : B \to \mathbb{P}^1$ intersect each fiber f in a point of order two. This fact turned out to be crucial in [20] to construct commuting involutions on B. First, it follows from the commutativity of figure 5 that

$$\psi^{-1} \circ \Delta^{-1} = \kappa^{-1} \circ \pi^{-1} . \tag{4.17}$$

But, we have shown in section 3 that

$$\kappa^{-1} \circ \pi^{-1}(s) , \qquad \kappa^{-1} \circ \pi^{-1}(i) , \qquad \kappa^{-1} \circ \pi^{-1}(j) , \qquad \kappa^{-1} \circ \pi^{-1}(t'') \tag{4.18}$$

correspond to sections of $\beta : B \to \mathbb{P}^1$. Hence,

$$\psi^{-1} \circ \Delta^{-1}(s) , \qquad \psi^{-1} \circ \Delta^{-1}(i) , \qquad \psi^{-1} \circ \Delta^{-1}(j) , \qquad \psi^{-1} \circ \Delta^{-1}(t'') \tag{4.19}$$

correspond to sections of $\beta : B \to \mathbb{P}^1$. Secondly, recall from above that $\Delta^{-1}(s)$, $\Delta^{-1}(i)$, $\Delta^{-1}(j)$ and $\Delta^{-1}(t'')$ are in the branch locus of

$$\psi : B \to \bar{Q} \cong \frac{B}{(-1)_B} . \tag{4.20}$$

Hence, the sections (4.18) are point-wise fixed under $(-1)_B$. Of course, this has to hold for the section e_9 corresponding to $\kappa^{-1} \circ \pi^{-1}(s)$, since we had chosen e_9 to be the zero section. In addition, this proves the invariance under $(-1)_B$ of the sections e_6 and e_4 corresponding to $\kappa^{-1} \circ \pi^{-1}(i)$ and $\kappa^{-1} \circ \pi^{-1}(j)$ respectively. Hence, these sections intersect each fiber f at a point of order two. Finally, since $e_4 \dot{+} e_6$ is the last possible section invariant under $(-1)_B$, this proves that $e_4 \dot{+} e_6$ corresponds to $\kappa^{-1} \circ \pi^{-1}(t'')$.

Studying the branch locus of $\psi : B \to \bar{Q}$ in $\bar{Q}$, we see that the branch curve has six singularities. Hence, without resolving, B will have six A_1 type singularities at the pre-image of these points. Thus, each section e_9, e_4 and e_6 contains two of these singularities. This is consistent with the explanation in the previous section. Furthermore, the section $e_4 \dot{+} e_6$ contains all six singularities.

5. B as a ruled surface

In this section, we will show that B is a ruled surface. A ruled surface is a two dimensional fibration whose generic fiber is not a torus T^2 but, rather, $\mathbb{P}^1$. Expressing B as a ruled surface will be very helpful in determining the transformation laws of the elements of $H_2(B,\mathbb{Z})$ under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group. Recall that B is elliptically fibered with respect to the mapping $\beta : B \to \mathbb{P}^1$. This mapping is defined by $sq \circ \beta = p_1 \circ \pi \circ \kappa$ where

$$sq \circ \beta : B \xrightarrow{\kappa} W_M \xrightarrow{\pi} Q \xrightarrow{p_1} \mathbb{P}^1 , \tag{5.1}$$

where $p_1 : Q \to \mathbb{P}^1$ is the natural projection of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ onto its first $\mathbb{P}^1$ factor.

Now recall that there is a second natural projection, namely, $p_2 : Q \to \mathbb{P}^1$ where $\mathbb{P}^1$ is the second $\mathbb{P}^1$ factor of Q. Clearly, then, we can construct a second fibration of B,

$$\delta : B \to \mathbb{P}^1, \tag{5.2}$$

where

$$\delta = p_2 \circ \pi \circ \kappa \tag{5.3}$$

and

$$\delta : B \xrightarrow{\kappa} W_M \xrightarrow{\pi} Q \xrightarrow{p_2} \mathbb{P}^1 . \tag{5.4}$$

What is the generic fiber of the mapping δ? To answer this, it is very helpful to use an alternative way that B covers Q, namely, the covering through the surface $\bar{Q} = \mathbb{P}^1 \times \mathbb{P}^1$ given in (4.10). This is expressed as the middle routing in figure 6. Using the commutativity of figure 5, we can write

$$\delta = p_2 \circ \Delta \circ \psi , \tag{5.5}$$

where now

$$\delta : B \xrightarrow{\psi} \bar{Q} \xrightarrow{\Delta} Q \xrightarrow{p_2} \mathbb{P}^1 . \tag{5.6}$$

We showed in the previous section that for a generic $x \in \mathbb{P}^1_2 \subset Q$, $\Delta^{-1}(p_2^{-1}(x)) \cong \mathbb{P}^1$. Furthermore, note that if we identify $\mathbb{P}^1_2 \subset Q$ with $\mathbb{P}^1_2 \subset \bar{Q}$ then $\bar{p}_2^{-1}(x) = \Delta^{-1}(p_2^{-1}(x))$. For a generic point $x \in \mathbb{P}^1_2 \subset \bar{Q}$, the fiber $\bar{p}_2^{-1}(x)$ intersects the branch locus of ψ in $\bar{Q}$ only at two points, each contained in the bidegree $(2,1)$ curve $\Delta^{-1}(t'')$. This is indicated by the dashed line in figure 7. Therefore, the double cover $\psi^{-1}(\bar{p}_2^{-1}(x))$ of $\Delta^{-1}(t'')$ is two $\mathbb{P}^1$ lines identified at two points and, hence, is isomorphic to $\mathbb{P}^1$. But

$$\mathbb{P}^1 \cong \psi^{-1}(\bar{p}_2^{-1}(x)) = (p_2 \circ \Delta \circ \psi)^{-1}(x) = \delta^{-1}(x) \tag{5.7}$$

is the generic fiber of (5.2). We conclude that B can be written as the fibration $\delta : B \to \mathbb{P}^1$ whose generic fiber is $\mathbb{P}^1$ and, hence, B is a ruled surface. Note that the fiber of δ is a degree four cover of the fiber of p_2 or, equivalently, a degree two cover of the fiber of $\bar{p}_2$. Adopting the second point of view will simplify many calculations.

What are the singular fibers of $\delta : B \to \mathbb{P}^1$? First consider the three bidegree $(0,1)$ curves $\Delta^{-1}(s)$, $\Delta^{-1}(i)$ and $\Delta^{-1}(j)$ in the branch locus of ψ in $\bar{Q}$. These are shown in figure 7, where we have defined $\bar{p}_2(\Delta^{-1}(s)) = a$, $\bar{p}_2(\Delta^{-1}(i)) = b$ and $\bar{p}_2(\Delta^{-1}(j)) = c$. Note

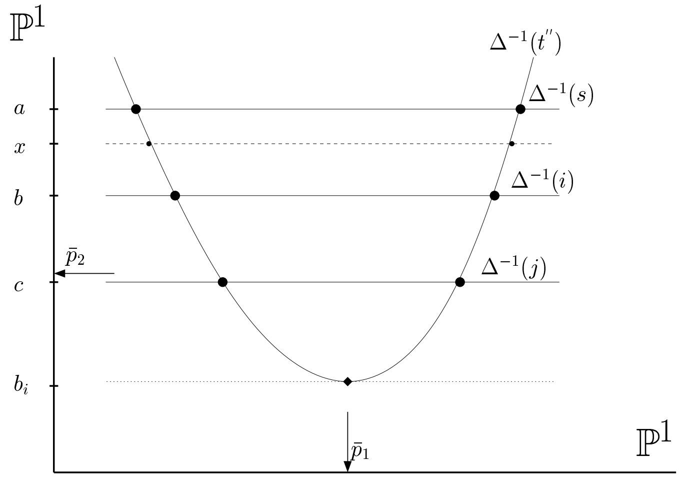

Figure 7: A schematic representation of the branch locus of ψ in $\bar{Q} = \mathbb{P}^1 \times \mathbb{P}^1$. In addition, a fiber over a generic point x represented by a dashed line and a fiber over a branch point b_i are shown.

that each of these curves is a $\mathbb{P}^1$ that intersects $\Delta^{-1}(t'')$ at two points. Hence, the pre-image under ψ in B of each of these points must be blown-up to obtain a smooth surface. It follows that the pre-image under ψ of each $(0,1)$ curve is a double $\mathbb{P}^1$ line intersecting two exceptional divisors in two distinct points. As discussed previously, we can identify these fibers as

$$e \cup (n_1 \cup n_2) = \delta^{-1}(a)\,, \qquad e_6 \cup (n_3 \cup n_4) = \delta^{-1}(b)\,, \qquad e \cup (n_5 \cup n_6) = \delta^{-1}(c) \tag{5.8}$$

respectively. They are shown explicitly in figure 8.

Are there other singular fibers of $\delta : B \to \mathbb{P}^1$? Consider the $(2,1)$ curve $\Delta^{-1}(t'')$ in $\bar{Q}$. Clearly, under the projection $\bar{p}_1 : \bar{Q} \to \mathbb{P}^1_1$ this curve is a degree one cover of $\mathbb{P}^1$ and, hence,

$$\chi(\Delta^{-1}(t'')) = \chi(\mathbb{P}^1) = 2\,. \tag{5.9}$$

However, under the projection $\bar{p}_2 : \bar{Q} \to \mathbb{P}^1$, $\Delta^{-1}(t'')$ is a degree two cover of $\mathbb{P}^1$. Then the Riemann-Hurwitz formula states that

$$\chi(\Delta^{-1}(t'')) = 2\chi(\mathbb{P}^1 \setminus \{b_i\}) + A\,, \tag{5.10}$$

where $\{b_i\}$ for $i = 1, \dots, A$ are the branch points of the simple ramification points of $\Delta^{-1}(t'')$ over $\mathbb{P}^1_2$. By definition, above each $b_i \in \mathbb{P}^1_2$, $\Delta^{-1}(t'')$ intersects $\bar{p}_2^{-1}(b_i)$ in only one point, not in two. At this point, the ramification point R_i, the fiber $\bar{p}_2^{-1}(b_i)$ intersects $\Delta^{-1}(t'')$ transversally. This is shown in figure 7. Therefore, the pre-image under ψ of each of $\bar{p}_2^{-1}(b_i)$ consists of two distinct $\mathbb{P}^1$ lines in B intersecting in a point. Note, however, that this point is smooth in B. How many such fibers are there? Combining equations (5.9) and (5.10) we find

$$A = 2\,. \tag{5.11}$$

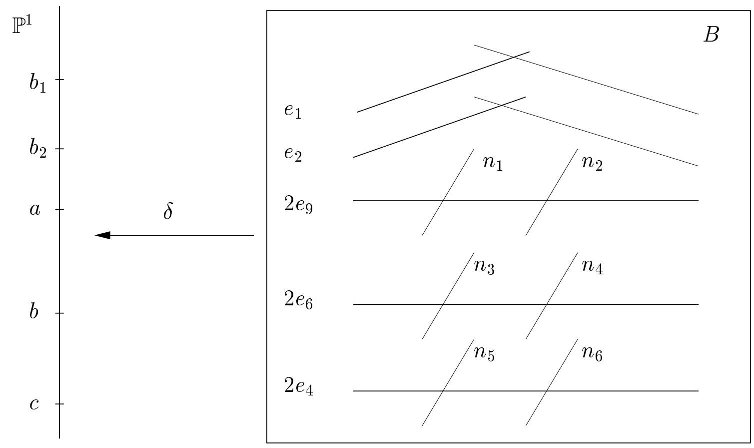

Figure 8: The singular fibers of $\delta : B \to \mathbb{P}^1$.

We conclude that there are two such singular fibers, $\delta^{-1}(b_i)$ for $i = 1, 2$. Although not immediately obvious, we will show in the next section that one $\mathbb{P}^1$ component of $\delta^{-1}(b_1)$ can be identified with e_1, whereas one $\mathbb{P}^1$ component of $\delta^{-1}(b_2)$ is e_2. These two singular fibers are described in figure 8.

In summary, we have shown that, in addition to the fibration $\beta : B \to \mathbb{P}^1$ discussed in [20] and section 3, there is a second way to fiber B, namely, $\delta : B \to \mathbb{P}^1$. The δ fibers are generically $\mathbb{P}^1$ and, hence, B is a ruled surface. This second fibration has five singular fibers, all shown explicitly in figure 8. This fibration will be very helpful in identifying the set of generators of $H_2(B, \mathbb{Z})$, which will be done in the next section.

6. Generators of $H_2(B, \mathbb{Z})$

Recall that a generic rational elliptic surfaces B is a $\mathbb{P}^2$ blown-up by a $\mathbb{P}^1$ at each of nine separate points. Furthermore, each of these $\mathbb{P}^1$ blown-up curves is an irreducible, exceptional divisor of B with self-intersection -1. We denote these nine exceptional divisors by $e_1, \ldots, e_9$, and have shown that $e_1, \ldots, e_9$ and l, where l is the pre-image of a line in $\mathbb{P}^2$, form a set of generators of $H_2(B, \mathbb{Z})$. However, the rational elliptic surfaces B that we wish to consider are not generic. Rather, they form a restricted two-dimensional sub-family of such surfaces. One can show that, in this case, some of the nine points in $\mathbb{P}^2$ coalesce. This greatly obscures the construction of a surface B in this two dimensional sub-family as a blow-up of a $\mathbb{P}^2$. In fact, as argued in [20] and section 3, it is more convenient to obtain these surfaces by different means, namely, as the four-fold cover of Q with a specific branch locus. Be that as it may, for surfaces in this two dimensional sub-family it is now essential that we exactly express B as a blow-up of $\mathbb{P}^2$. In doing this, we will explicitly identify nine divisors with self-intersection -1, which we again denote as $e_1, \ldots, e_9$, and give their structure in B. These nine divisors, along with the pre-image of a line l in $\mathbb{P}^2$, will again form the set of generators of $H_2(B, \mathbb{Z})$. To proceed, we find it essential to use the $\mathbb{P}^1$ fibration $\delta : B \to \mathbb{P}^1$ presented in the previous section.

To identify the rational elliptic surfaces B as a blow-up of $\mathbb{P}^2$, we need to invoke the so-called Castelnuovo-Enriques criterion. This involves a process called blow-down, the inverse of blowing-up. The Castelnuovo-Enriques criterion states that for any surface containing a -1 curve, that is, a curve isomorphic to $\mathbb{P}^1$ of self-intersection -1, one can blow down this curve and obtain a smooth surface. For more details on the Castelnuovo-Enriques criterion see [25]. To apply this process to the $d\mathbb{P}_9$ surfaces of our two dimensional sub-family, we must identify nine -1 curves and successively blow them down. To do this, recall that the surface B can be considered as a ruled surface, that is

$$\delta : B \to \mathbb{P}^1 \tag{6.1}$$

with the generic fiber of δ being isomorphic to $\mathbb{P}^1$. We will denote the class of such a fiber by F. As discussed in section 5, this fibration has five singular fibers, each graphically described in figure 8. First, we consider the two singular fibers each of which consists of two $\mathbb{P}^1$ components which intersect in a single point. Choose one of these two fibers and denote its components by c_1 and c_2. Then, as homology classes

$$F = c_1 + c_2\,. \tag{6.2}$$

Note that here and henceforth, $+$ and $-$ denote the group operations in the homology group $H_2(B, \mathbb{Z})$. These must not be confused with the addition and inverse of elements in $\Gamma(B)$, defined in section 3, which, to distinguish them, were denoted using $\dot{+}$ and $\dot{-}$. Recalling that for any fiber class $F^2 = 0$, and using $c_1 \cdot c_2 = 1$, we find

$$0 = F^2 = (c_1 + c_2)^2 = c_1^2 + 2 + c_2^2\,. \tag{6.3}$$

Since the components c_1 and c_2 are completely symmetric in B, it follows that

$$c_1^2 = c_2^2 = -1\,. \tag{6.4}$$

Therefore, this singular fiber contains two -1 curves. By the Castelnuovo-Enriques criterion explained above, we can blow down either of them. Choose one component, for example c_1, and shrink it to a point. What is the self-intersection of the remaining component? Since this component now forms a complete fiber of the fibration induced by δ on the blown down surface, its self-intersection is zero. Hence, there is no longer any -1 curve in this fiber. Of course, the second fiber of $\delta : B \to \mathbb{P}^1$ consisting of two $\mathbb{P}^1$ components intersecting at a point did not get changed in the process of blowing down. Therefore, it still admits two -1 curves, one of which can now be blown down. Hence, we have identified two -1 curves in B which can be blown down simultaneously.

We now show that there are additional -1 curves in the remaining three singular fibers of $\delta : B \to \mathbb{P}^1$. Start with the fiber containing the double component e_9 and the two exceptional divisors n_1 and n_2. For a pictorial representation, we refer the reader to figure 8. For this fiber we can write

$$F = 2e_9 + n_1 + n_2\,. \tag{6.5}$$

Recall that we obtained n_1 and n_2 by blowing up two A_1 surface singularities. Hence, blowing them down would not lead to a smooth surface. Therefore, by the Castelnuovo-Enriques criterion, neither exceptional divisor can be a -1 curve. It can be shown, however, that, since they arise by blowing up A_1 surface singularities, their self-intersection number is -2. Using this, and the fact that e_9 intersects each of n_1 and n_2 in one point, we find that

$$0 = (2e_9 + n_1 + n_2)^2 = 4e_9^2 - 2 - 2 + 2(2+2) \tag{6.6}$$

and, hence

$$e_9^2 = -1\,. \tag{6.7}$$

Therefore, we can blow down e_9 and obtain a smooth surface. What are the self-intersection numbers of the images of n_1 and n_2 in the blown down surface? Since this induced fiber in the blown down surface consists of two $\mathbb{P}^1$ components which intersect at one point, we are in the same situation as described above and can conclude that the image of each of n_1 and n_2 in the blown down surface self-intersects as -1. Hence, we can pick one of them and blow it down.

Of course, the reasoning applied to the fiber $2e_9 + n_1 + n_2$ can be applied to the fibers $2e_6+n_3+n_4$ and $2e_4+n_5+n_6$. Therefore, the three singular fibers each containing a double component and two exceptional divisors allow us to blow down six -1 curves. Taking the previous two singular fiber of $\delta : B \to \mathbb{P}^1$ into account as well, we have found eight -1 curves which can be simultaneously blown down. But since $B \cong d\mathbb{P}_9$, we need to find one additional -1 curve to obtain a $\mathbb{P}^2$. Here, we will simply state a result proven in [21]. Recall the map

$$\psi : B \to \bar{Q}\,, \tag{6.8}$$

which is a double cover branched over the three $(0,1)$ rulings $\Delta^{-1}(s)$, $\Delta^{-1}(i)$, $\Delta^{-1}(j)$ and the $(2,1)$ curve $\Delta^{-1}(t'')$. Consider a $(1,1)$ curve in $\bar{Q}$ passing through one of the two intersection points of each of $\Delta^{-1}(s) \cap \Delta^{-1}(t'')$, $\Delta^{-1}(i) \cap \Delta^{-1}(t'')$ and $\Delta^{-1}(j) \cap \Delta^{-1}(t'')$. Specifically, we choose this curve to pass through the points whose pre-image under ψ blows-up to n_2, n_4 and n_6 respectively. It can be shown that such a curve in $\bar{Q}$ exists and that its proper transform in B is a -1 curve which is a section of the fibration $\delta : B \to \mathbb{P}^1$. We denote this -1 curve by e_7.

We are finally in a position to construct our restricted surface B as a blow-up of $\mathbb{P}^2$. In the process, we will specifically identify, and give the properties of, nine -1 curves in B. To do this, begin with one such surface B. First, consider the two singular fibers of $\delta : B \to \mathbb{P}^1$ each consisting of two components. Denote by e_1 the component of one of these fibers which does not intersect e_7 and by e_2 the component of the other fiber which does not intersect e_7. Blow them down. Next, blow the -1 curves e_9, e_6 and e_4 down. Continuing, blow the images of the curves n_1, n_3 and n_5 down. Note that none of the curves e_9, e_6, e_4, n_1, n_3 or n_5 intersects e_7. Finally, upon blowing down e_7 we obtain $\mathbb{P}^2$. Of the curves $e_1, e_2, e_4, e_6, e_7, e_9, n_1, n_3$ and n_5, all are -1 curves with the exception of n_1, n_3 and n_5. In addition, some of the e-type and n-type curves intersection This motivates us to define

$$e_3 = e_4 + n_5\,, \qquad e_5 = e_6 + n_3\,, \qquad e_8 = e_9 + n_1\,. \tag{6.9}$$

It is straightforward to verify that each of e_3, e_5 and e_8 are -1 curves and that $e_1, \ldots, e_9$ are all non-intersecting. To reiterate, if we start with a restricted surface B and blow down the curves $e_1, e_2, e_4, e_6, e_9, n_1, n_3, n_5$ and e_7 we obtain a $\mathbb{P}^2$. Reversing the process, we have constructed B as the blow-up of a $\mathbb{P}^2$ surface, as desired.

If we denote by l the pull back of a curve on $\mathbb{P}^2$, then $e_1, \ldots, e_9$ and l form a set of generators for $H_2(B, \mathbb{Z})$ with the same intersection numbers as given in (4.1). That is,

$$H_2(B, \mathbb{Z}) = \mathbb{Z}l \oplus (\oplus_{i=1}^{9} \mathbb{Z}e_i) . \tag{6.10}$$

Therefore, we have successfully described a set of generators of the curve homology of B in terms of the nine -1 curves and the pre-image of a $\mathbb{P}^1$ line in $\mathbb{P}^2$. Note that this motived our original choice in [20] and section 3 to name the two non-zero sections of order two as e_4 and e_6.

In this basis, the components of the I_2 fibers can be written as

$$\begin{aligned}
n_1 &= e_8 - e_9 \\
o_1 &= f - e_8 + e_9 \\
n_2 &= l - e_7 - e_8 - e_9 \\
o_2 &= 2l - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 \\
n_3 &= e_5 - e_6 \\
o_3 &= f - e_5 + e_6 \\
n_4 &= l - e_7 - e_6 - e_5 \\
o_4 &= 2l - e_1 - e_2 - e_3 - e_4 - e_8 - e_9 \\
n_5 &= e_3 - e_4 \\
o_5 &= f - e_3 + e_4 \\
n_6 &= l - e_7 - e_4 - e_3 \\
o_6 &= 2l - e_1 - e_2 - e_5 - e_6 - e_8 - e_9 .
\end{aligned} \tag{6.11}$$

where f is the fiber class of the elliptic fibration $\beta : B \to \mathbb{P}^1$. Let us make a remark on these equations. The expressions for n_1, n_3 and n_5 follow easily from (6.9). To understand the expressions for n_2, n_4 and n_6, observe the following. Take, for example, n_2. It is the component of the singular fiber of $\delta : B \to \mathbb{P}^1$ intersecting e_7 which is left after blowing down e_9, e_7 and n_1. Hence, the image of n_2 in $\mathbb{P}^2$ must be l. It follows that n_2 in B is given by l reduced by the exceptional divisors $2e_9$, e_7, and n_1. That is

$$n_2 = l - 2e_9 - e_7 - n_1 , \tag{6.12}$$

which is the expression given in (6.11). The equations for n_4 and n_6 are derived in a similar way. Once we have defined $n_1, \ldots, n_6$, the proper transforms $o_1, \ldots, o_6$ are given by $o_i = f - n_i$, $i = 1, \ldots, 6$. Recall from section 3 that the fiber class f is $f = 3l - \sum_{i=1}^{9} e_i$. Note that the section e_7 intersects n_2, n_4, n_6 and o_1, o_3, o_5. Finally, even through $e_1, \ldots, e_9$ and l form a canonical set of generators, the set e_1, e_2, e_4, e_6, e_7, e_9, n_1, n_3, n_5 and l also generates $H_2(B, \mathbb{Z})$. We will use these two sets of generators interchangeably depending on the context.

JHEP01(2004)059

7. Transformations under $(-1)_B$

In the following sections, we will determine the transformation laws of the elements of $H_2(B,\mathbb{Z})$ under the involutions τ_{B1} and τ_{B2} defined in section 3. To determine these transformations, we will use both the elliptic fibration $\beta : B \to \mathbb{P}^1$, whose fiber we continue to call f, and the ruling $\delta : B \to \mathbb{P}^1$, whose fiber we continue to call F. We will also need all three descriptions of B shown in figure 6. In this section, for completeness and to establish methodology, we will determine the induced action of $(-1)_B$ on $H_2(B,\mathbb{Z})$.

To begin, observe that

$$(-1)_B(f) = f\,. \tag{7.1}$$

This is clear since $(-1)_B$ acts fiber-wise on the elliptic fibration. In addition, note that

$$(-1)_B(F) = F\,. \tag{7.2}$$

This follows from the fact that $(-1)_B$ is the covering involution for $\psi : B \to \bar{Q}$ and $F = \delta^{-1}(x)$ is the double cover of the $\mathbb{P}^1$ line $\bar{p}_2^{-1}(x)$, where $\bar{p}_2$ is the second natural projection of $\bar{Q}$, namely, $\bar{p}_2 : \bar{Q} \to \mathbb{P}^1_2$.

Now consider the two singular fibers of $\delta : B \to \mathbb{P}^1$ containing the sections e_1 and e_2 respectively. Choose one of them, say the fiber containing e_1. As discussed above, this fiber can be written as

$$F = e_1 + c_2\,, \tag{7.3}$$

where c_2 intersects e_7. Since this fiber arises as the pre-image under ψ of a single $\mathbb{P}^1$ line which intersects the branch locus of ψ in $\bar{Q}$ at a single point, it follows that e_1 and c_2 must be exchanged under $(-1)_B$. That is,

$$c_2 = (-1)_B(e_1)\,. \tag{7.4}$$

Exactly the same statement is true for the singular fiber containing e_2. Therefore, the homology class of the fiber F can be written as

$$F = e_i + (-1)_B(e_i)\,, \qquad i = 1, 2\,. \tag{7.5}$$

Note that this is consistent with the expression (7.2). Now consider a generic fiber of $\delta : B \to \mathbb{P}^1$. On might imagine that l would be the class of this fiber. However, it is clear from the previous discussion that the generic fiber must intersect e_7 at a single point. Using the intersections $l \cdot e_7 = 0$ and $e_7^2 = -1$, we see that the fiber class can be written as

$$F = l - e_7\,. \tag{7.6}$$

Note that it follows from (7.2) that

$$(-1)_B(l - e_7) = l - e_7\,. \tag{7.7}$$

From (7.5) and (7.6), we conclude that

$$(-1)_B(e_i) = l - e_7 - e_i\,, \qquad i = 1, 2\,. \tag{7.8}$$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_8	l
$(-1)_B$	$l - e_7 - e_1$	$l - e_7 - e_2$	e_3	e_4	e_5	e_6	$l - e_1 - e_2$	e_8	e_9	$2l - (e_1 + e_2 + e_7)$

Table 1: The action of $(-1)_B$ on the canonical set of generators of $H_2(B, \mathbb{Z})$.

Next consider the six exceptional divisors n_i, $i = 1, \ldots, 6$. Since the image under ψ of the six surface singularities in $\bar{B}$ are contained in the branch locus of ψ in $\bar{Q}$, it follows that the six exceptional divisors n_i, $i = 1, \ldots, 6$ are fixed under $(-1)_B$. That is

$$(-1)_B(n_i) = n_i\,, \qquad i = 1, \ldots, 6\,. \tag{7.9}$$

Recall that the proper transforms are defined to be $o_i = f - n_i$ for $i = 1, \ldots, 6$. Then, it follows from (7.1) and (7.9) that

$$(-1)_B(o_i) = o_i\,, \qquad i = 1, \ldots, 6\,. \tag{7.10}$$

The sections e_4, e_6 and e_9 are invariant under $(-1)_B$ by definition. Applying these results to the sections e_3, e_5 and e_8 defined in (6.9) using the linearity of $(-1)_B$, the invariance of e_4, e_6 and e_9 and (7.9), we find, for example, that

$$(-1)_B(e_3) = (-1)_B(e_4 + n_5) = e_4 + n_5 = e_3\,. \tag{7.11}$$

Similarly, it follows that

$$(-1)_B(e_5) = e_5\,, \qquad (-1)_B(e_8) = e_8\,. \tag{7.12}$$

Finally, using o_2 given in (6.11), the fact that $(-1)_B(o_2) = o_2$ and the invariance of e_3, e_4, e_5 and e_6 under $(-1)_B$, it follows that

$$(-1)_B(l) = 2l - (e_1 + e_2 + e_7)\,. \tag{7.13}$$

Furthermore, from (7.7) and (7.13) we find

$$(-1)_B(e_7) = l - (e_1 + e_2)\,. \tag{7.14}$$

This completes the analysis for the action of $(-1)_B$ on the set of generators $e_1, \ldots, e_9$ and l of $H_2(B, \mathbb{Z})$. These results are summarized in table 1. The transformation of an arbitrary class in $H_2(B, \mathbb{Z})$ follows immediately from these results.

8. The action of α_B

In this section, we begin our analysis of the transformations of the elements of $H_2(B, \mathbb{Z})$ under τ_{B1} and τ_{B2}. Recalling from (3.33) that $\tau_{B1} = t_{e_6} \circ \alpha_B$ and $\tau_{B2} = t_{e_4} \circ \alpha_B$, we start by finding the action of α_B on $H_2(B, \mathbb{Z})$. First, observe that

$$\alpha_B(f) = f\,. \tag{8.1}$$

This follows from the fact that α_B preserves the fiber of the elliptic fibration $\beta : B \to \mathbb{P}^1$. The commutativity of figure 6 implies that $\delta^{-1}(x) = \kappa^{-1}(\pi^{-1}(p_2^{-1}(x)))$. Since α_B is the covering involution for map κ, this shows that the fiber class of $\delta : B \to \mathbb{P}^1$ is also invariant under α_B. That is,

$$\alpha_B(F) = F\,. \tag{8.2}$$

Let us first discuss the generic representation of F given in (7.6). Then this and (8.2) imply that under α_B

$$\alpha_B(l - e_7) = l - e_7\,. \tag{8.3}$$

Now, consider the singular fiber $2e_9 + n_1 + n_2$ of $\delta : B \to \mathbb{P}^1$. Note that $\alpha_{\bar{B}}$ exchanges the two surface singularities in $\bar{B}$ which blow up into n_1 and n_2. Hence, α_B exchanges n_1 and n_2. That is,

$$\alpha_B(n_1) = n_2\,, \qquad \alpha_B(n_2) = n_1\,. \tag{8.4}$$

Since, by (8.2), $2e_9 + n_1 + n_2$ must be invariant under α_B, it follows that

$$\alpha_B(e_9) = e_9\,. \tag{8.5}$$

Similar remarks hold for the singular fibers $2e_6 + n_3 + n_4$ and $2e_4 + n_5 + n_6$. We see, therefore, that

$$\alpha_B(n_3) = n_4\,, \qquad \alpha_B(n_4) = n_3 \tag{8.6}$$

and

$$\alpha_B(n_5) = n_6\,, \qquad \alpha_B(n_6) = n_5\,. \tag{8.7}$$

Combining this with the invariance of both singular fibers under α_B, we conclude that

$$\alpha_B(e_6) = e_6\,, \qquad \alpha_B(e_4) = e_4\,. \tag{8.8}$$

Next consider the proper transform $o_i = f - n_i$ for $i = 1, \ldots, 6$. Then it follows from (8.1), (8.4), (8.6) and (8.7) that

$$\alpha_B(o_{2j-1}) = o_{2j}\,, \qquad \alpha_B(o_{2j}) = o_{2j-1} \tag{8.9}$$

for $j = 1, 2, 3$. It is easy to read off the transformation laws for e_3, e_5 and e_8 defined in (6.9). For example, using (8.7), (8.8) and (6.11) we see that

$$\alpha_B(e_3) = \alpha_B(e_4 + n_5) = e_4 + n_6 = l - e_3 - e_7\,. \tag{8.10}$$

Similarly, we have

$$\alpha_B(e_5) = \alpha_B(e_6 + n_3) = e_6 + n_4 = l - e_5 - e_7 \tag{8.11}$$

and

$$\alpha_B(e_8) = \alpha_B(e_9 + n_1) = e_9 + n_2 = l - e_7 - e_8\,. \tag{8.12}$$

Let us now discuss the transformation law of the sections e_1 and e_2. Each of these is contained in one of the two singular fibers of $\delta : B \to \mathbb{P}^1$ consisting of two components. Consider either one of these singular fibers. Since it must be stable under α_B, α_B can either exchange its two components or leave them invariant. To answer this question, we must analyze these two fibers of δ more closely. By definition, these fibers are double covers of the $(0,1)$ rulings in $\bar{Q}$ which intersect the $(2,1)$ curve $\Delta^{-1}(t^{''})$ tangentially. It is easy to see that these $(0,1)$ rulings in $\bar{Q}$ are double covers of the two $(0,1)$ rulings in Q with pass through either $r^{'} \cap t^{''}$ or $r \cap t^{''}$. Let us assume that e_1 is contained in the fiber of δ corresponding to the degree four cover of the $(0,1)$ ruling in Q which passes through the point $r^{'} \cap t^{''}$. It follows that e_2 is contained in the fiber of δ corresponding to the degree four cover of the $(0,1)$ ruling in Q which passes through the point $r \cap t^{''}$. From the commutativity of figure 6, we can discuss these two fibers of δ using the relation $\delta^{-1}(x) = \kappa^{-1}(\pi^{-1}(p_2^{-1}(x)))$. To determine the transformation law of e_1, consider the $(0,1)$ ruling in Q which passes through $r^{'} \cap t^{''}$. This ruling intersects the branch divisor of the map π at two different points. Hence, its double cover in W_M is a smooth $\mathbb{P}^1$. This curve intersects the branch divisor of κ at a single point in f_0 in W_M. Therefore, its pre-image in B consists of two components which are exchanged by the covering involution for κ and, hence, by α_B. Since one of the two components is e_1,

$$F = e_1 + \alpha_B(e_1) \tag{8.13}$$

represents the class of a full fiber of δ. It follows from this and (7.6) that

$$e_1 + \alpha_B(e_1) = l - e_7\,, \tag{8.14}$$

or equivalently

$$\alpha_B(e_1) = l - e_1 - e_7\,. \tag{8.15}$$

What about the fiber of δ containing e_2? Consider the $(0,1)$ ruling in Q which passes through $r \cap t^{''}$. This is the only point where this ruling intersects the branch locus of π in Q. Hence, the pre-image of this ruling in W_M consists of two components, both isomorphic to $\mathbb{P}^1$. What is the pre-image of these two $\mathbb{P}^1$ lines in B? Since each component intersects the branch locus of κ in two points, namely its intersection with f_0 and the point $\pi^{-1}(p_1)$ in W_M, the pre-image of each component is a smooth $\mathbb{P}^1$. Clearly, these components are not exchanged under the covering involution of κ and, hence, under α_B. One of the components we called e_2. Therefore,

$$\alpha_B(e_2) = e_2\,. \tag{8.16}$$

Finally, solving the equations $\alpha_B(o_1) = o_2$ and $\alpha_B(l - e_7) = l - e_7$ using (3.8), (6.11), (8.8), (8.10), (8.11), (8.15) and (8.16), we obtain

$$\alpha_B(l) = 3l - e_1 - e_3 - e_5 - 2e_7 - e_8 \tag{8.17}$$

and

$$\alpha_B(e_7) = 2l - e_1 - e_3 - e_5 - e_7 - e_8\,. \tag{8.18}$$

This concludes our calculation of the transformation laws of the canonical set of generators of $H_2(B, \mathbb{Z})$ under α_B. The results are summarized in table 2. The action of α_B on any class in $H_2(B, \mathbb{Z})$ follows immediately from these results.

	e_1	e_2	e_3	e_4	e_5
α_B	$l-e_7-e_1$	e_2	$l-e_3-e_7$	e_4	$l-e_5-e_7$
	e_6	e_7	e_8	e_9	l
α_B	e_6	$2l-(e_1+e_3+e_5+e_7+e_8)$	$l-e_7-e_8$	e_9	$3l-(e_1+e_3+e_5+2e_7+e_8)$

Table 2: The action of α_B on the canonical set of generators of $H_2(B,\mathbb{Z})$.

9. The action of t_{e_6} and t_{e_4}

In this section, we complete the analysis of the transformations of the elements of $H_2(B,\mathbb{Z})$ under τ_{B1} and τ_{B2} by computing the actions of t_{e_6} and t_{e_4}. We will proceed in several steps, first computing the transformations of the I_2 fiber components n_i and o_i for $i=1,\dots,6$ and then using these results to find the transformations of the canonical set of generators of $H_2(B,\mathbb{Z})$.

9.1 The action of t_{e_6} and t_{e_4} on fiber components

Recall that t_{e_6} and t_{e_4} preserve the fibers of the elliptic fibration $\beta: B\to\mathbb{P}^1$. That is

$$t_{e_6}(f)=f\,,\qquad t_{e_4}(f)=f\,. \tag{9.1}$$

As discussed in section 3, any surface B in the restricted two parameter family has six single I_2 fibers. Each such I_2 fiber is composed of two $\mathbb{P}^1$ components, n and o, intersecting at two singular points. This is shown in figure 3. That is, the I_2 singular fibers of B can be written as $n_i\cup o_i$, $i=1,\dots,6$. The relationship of these fiber components to the canonical set of generators $e_1,\dots,e_9$ and l of $H_2(B,\mathbb{Z})$ was given in (6.11). It follows from these remarks and (9.1) that the action of t_{e_6} and t_{e_4} either exchanges the two components of an I_2 fiber or leaves them invariant. The remainder of this sub-section is devoted to deciding which of these possibilities occur.

We begin our analysis with the first I_2 fiber, defined as $n_1\cup o_1$, and consider the action on it of t_{e_6}. Recall that each smooth fiber of $\beta: B\to\mathbb{P}^1$ admits an abelian group structure whose identity element is the point of intersection of the fiber with the zero section $e=e_9$. It can be shown that, after removing the two singular points, each I_2 fiber also carries an abelian group structure. Using (6.11) and the intersection numbers of the canonical set of generators of $H_2(B,\mathbb{Z})$, we see that

$$n_1\cdot e=1\,,\qquad o_1\cdot e=0\,. \tag{9.2}$$

Hence, the identity element is in the n_1 component. In addition, note that

$$n_1\cdot e_6=0\,,\qquad o_1\cdot e_6=1\,. \tag{9.3}$$

These results are represented pictorially in figure 9. Now consider a specific point on n_1, which we choose to be the identity element $n_1\cap e$. Translating this point by t_6 gives

$$t_6(n_1\cap e)=(n_1\cup o_1)\cap e_6=o_1\cap e_6\,, \tag{9.4}$$

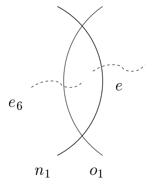

Figure 9: A schematic representation of the intersection of e and e_6 with n_1 and o_1.

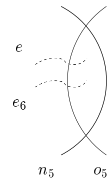

Figure 10: A schematic representation of the intersection of e and e_6 with n_5 and o_5.

which is a point in o_1. If follows the t_{e_6} must exchange n_1 and o_1. That is

$$t_{e_6}(n_1) = o_1\,, \qquad t_{e_6}(o_1) = n_1\,. \tag{9.5}$$

It is straightforward to see that t_{e_6} will exchange the components of any I_2 fiber for which e and e_6 intersect different components, as in figure 9.

What happens for an I_2 fiber for which e and e_6 intersect the same component? As an example, consider the fifth I_2 fiber defined as $n_5 \cup o_5$. Using (6.11) and the intersection numbers of the canonical set of generators of $H_2(B, \mathbb{Z})$, it is easy to show that

$$n_5 \cdot e = n_5 \cdot e_6 = 0 \tag{9.6}$$

and

$$o_5 \cdot e = o_5 \cdot e_6 = 1\,. \tag{9.7}$$

Here, the identity element is in the o_5 component. These results are represented in figure 10. Again, consider the zero element $o_5 \cap e$. Translating this point by t_{e_6} gives

$$t_{e_6}(o_5 \cap e) = (n_5 \cup o_5) \cap e_6 = o_5 \cap e_6\,, \tag{9.8}$$

which remains a point in o_5. We conclude that t_{e_6} leaves n_5 and o_5 stable. That is

$$t_{e_6}(n_5) = n_5\,, \qquad t_{e_6}(o_5) = o_5\,. \tag{9.9}$$

Clearly, this will be the case for any I_2 fiber for which e and e_6 intersect the same component, is in figure 10.

It is easy now to derive the remaining transformation laws of n_i and o_i for $i = 1, \ldots, 6$ under t_{e_6} and t_{e_4}. From (6.11) one can read off the intersection numbers of the various I_2 fiber components with e, e_6 and e_4. By the above remarks, it is then straightforward to determine the transformation laws of the fiber components under t_{e_6} and t_{e_4}. We summarize our findings in table 3, where we also specify the action of the compositions $\tau_{B1} = t_{e_6} \circ \alpha_B$ and $\tau_{B2} = t_{e_4} \circ \alpha_B$ on n_i and o_i for $i = 1, \ldots, 6$. To do the latter, we combine the results of this section with those of section 8.

9.2 Transformations of the canonical generators under t_{e_6}

Having found the action of t_{e_6} and t_{e_4} on the fiber components, we now use these results to compute the transformation laws of the canonical set of generators of $H_2(B, \mathbb{Z})$ under each translation. In this sub-section, we consider the action of t_{e_6}. Some of these laws are easy

	e_9	e_6	e_4	t_{e_6}	t_{e_4}	α_B	$\tau_{B1} = t_{e_6} \circ \alpha_B$	$\tau_{B2} = t_{e_4} \circ \alpha_B$
n_1	1	0	0	o_1	o_1	n_2	o_2	o_2
n_2	1	0	0	o_2	o_2	n_1	o_1	o_1
n_3	0	1	0	o_3	n_3	n_4	o_4	n_4
n_4	0	1	0	o_4	n_4	n_3	o_3	n_3
n_5	0	0	1	n_5	o_5	n_6	n_6	o_6
n_6	0	0	1	n_6	o_6	n_5	n_5	o_5
o_1	0	1	1	n_1	n_1	o_2	n_2	n_2
o_2	0	1	1	n_2	n_2	o_1	n_1	n_1
o_3	1	0	1	n_3	o_3	o_4	n_4	o_4
o_4	1	0	1	n_4	o_4	o_3	n_3	o_3
o_5	1	1	0	o_5	n_5	o_6	o_6	n_6
o_6	1	1	0	o_6	n_6	o_5	o_5	n_5

Table 3: The intersection of the components of the singular fibers with the sections e, e_6 and e_4, and their transformation under the automorphisms t_{e_6}, t_{e_4}, α_B, τ_{B1} and τ_{B2}.

to read off, others are more complicated. Let us begin with the easy ones. First, it follows from the definition of a translation, and the fact that $e_9 = e$ is the zero section, that

$$t_{e_6}(e_9) = e_9 \dot{+} e_6 = e_6 \,. \tag{9.10}$$

Furthermore, since e_6 is a section of order two, we have

$$t_{e_6}(e_6) = e_6 \dot{+} e_6 = e_9 \,. \tag{9.11}$$

With help of (9.10), (9.11) and table 3, it is now straightforward to determine the transformation laws of e_5 and e_8. Using, in addition, (6.9), the definition of $o_i = f - n_i$ for $i = 1, \dots, 6$ and (6.11), we find that

$$t_{e_6}(e_5) = t_{e_6}(e_6 + n_3) = e_6 + o_3 = e_9 + f - n_3 = f - e_5 + e_6 + e_9 \tag{9.12}$$

and

$$t_{e_6}(e_8) = t_{e_6}(e_9 + n_1) = e_6 + o_1 = e_6 + f - n_1 = f + e_6 - e_8 + e_9 \,. \tag{9.13}$$

To find the transformation laws of e_1, e_2, e_4 and e_7 under t_{e_6}, however, we need to exploit the relationship between elements in $H_2(B, \mathbb{Z})$ and line bundles which, until now, was unnecessary. First, by Poincare duality, we have a canonical isomorphism

$$H_2(B, \mathbb{Z}) \cong H^2(B, \mathbb{Z}) \,. \tag{9.14}$$

Furthermore, it can be shown that on a simply connected smooth surface

$$H^2(B, \mathbb{Z}) \cong Pic(B) \,, \tag{9.15}$$

where $Pic(B)$ is the space of all line bundles up to isomorphism on B. Hence, we obtain

$$H_2(B, \mathbb{Z}) \cong Pic(B) \,. \tag{9.16}$$

More explicitly, for any element $\xi \in H_2(B, \mathbb{Z})$, we have

$$\xi \to \mathcal{O}_B(\xi) \, . \tag{9.17}$$

What is the relationship between an involution $\tau : B \to B$ on $H_2(B, \mathbb{Z})$ and its action on $Pic(B)$? To answer this, consider a class $\xi \in H_2(B, \mathbb{Z})$ and its associated line bundle $\mathcal{O}_B(\xi) \in Pic(B)$. If τ is an involution on B, then we can define the pull-back bundle, $\tau^*\mathcal{O}_B(\xi)$, under τ which satisfies the commutative diagram

$$\begin{tikzcd} \tau^*\mathcal{O}_B(\xi) \arrow[r, "\pi_2"] \arrow[d, "\pi'"'] & \mathcal{O}_B(\xi) \arrow[d, "\pi"] \\ B \arrow[r, "\tau"] & B \end{tikzcd} \tag{9.18}$$

Map π_2 takes the fiber $\pi'^{-1}(p) \subset \tau^*\mathcal{O}_B(\xi)$ to the fiber $\pi^{-1}(\tau(p)) \subset \mathcal{O}_B(\xi)$ for all points $p \in B$. As is, the bundle $\tau^*\mathcal{O}_B(\xi)$ is abstractly defined. However, one can identify $\tau^*\mathcal{O}_B(\xi)$ more concretely, as we now show. We will assume that the class in $H_2(B, \mathbb{Z})$ can be represented by a curve, which we also call ξ. This implies that $\mathcal{O}_B(\xi)$ has a section

$$s : B \to \mathcal{O}_B(\xi) \tag{9.19}$$

whose vanishing locus is on ξ, that is, $s(\xi) = 0$. Now consider the map

$$s' : B \to \tau^*\mathcal{O}_B(\xi) \tag{9.20}$$

defined by

$$\pi_2 \circ s' = s \circ \tau \, . \tag{9.21}$$

Then, it is clear that s' is a section of $\tau^*\mathcal{O}_B(\xi)$ satisfying

$$s'(\tau^{-1}(\xi)) = 0 \, . \tag{9.22}$$

Hence, the curve, or homology cycle, representing $\tau^*\mathcal{O}_B(\xi)$ is $\tau^{-1}(\xi)$. That is,

$$\tau^*\mathcal{O}_B(\xi) \cong \mathcal{O}_B(\tau^{-1}(\xi)) \cong \mathcal{O}_B(\tau(\xi)) \, , \tag{9.23}$$

where we have used the fact that, since τ is an involution, $\tau^{-1} = \tau$. We can now answer the above question definitely. The action $\xi \to \tau(\xi)$ of an involution on $H_2(B, \mathbb{Z})$ induces the action

$$\mathcal{O}_B(\xi) \to \mathcal{O}_B(\tau^{-1}(\xi)) \cong \mathcal{O}_B(\tau(\xi)) \tag{9.24}$$

on $Pic(B)$. We will use this relationship to determine the transformation laws of e_1, e_2, e_4 and e_7 under translations.

To do this, we first need to review some relevant facts about line bundles. Recall that B is an elliptic fibration $\beta : B \to \mathbb{P}^1$. We want to discuss the group structure of both the smooth elliptic fibers, T^2, and the space of sections, $\Gamma(B)$, in terms of line bundles on T^2 and B respectively. First, we consider a smooth elliptic curve T^2 with a fixed zero point, which we call e. Each point $p \in T^2$ determines a line bundle, namely

$$p \to \mathcal{O}_{T^2}(p) \, , \tag{9.25}$$

which is of degree one since it is defined by a single point. This specifies a map $T^2 \to Pic^1(T^2)$, where $Pic^1(T^2) \subset Pic(T^2)$ is the space of degree one line bundles on T^2 up to isomorphism. Note that since $Pic^1(T^2)$ is not closed under tensor product multiplication, it does not form a group. However, exploiting the fact that we have a marked point $e \in T^2$, we can, alternatively, define a map for each point p to a degree zero line bundle, namely

$$p \to \mathcal{O}_{T^2}(p - e)\,. \tag{9.26}$$

$\mathcal{O}_{T^2}(p-e)$ is of degree zero since it is defined as the difference of two points. This specifies a map $T^2 \to Pic^0(T^2)$, which can be shown to be an isomorphism. $Pic^0(T^2)$ is the space of all degree zero line bundles on T^2 up to isomorphism. It is clear that the bundle associated to the zero point e under this map is the trivial bundle. Furthermore, $Pic^0(T^2)$ is closed under tensor product multiplication and forms a group. Note that the map $T^2 \to Pic^0(T^2)$ is a group homomorphism. This means that for any given three points $p, p', p'' \in T^2$ which obey

$$p \dot{+} p' = p''\,, \tag{9.27}$$

we have

$$\mathcal{O}_{T^2}(p - e) \otimes \mathcal{O}_{T^2}(p' - e) \cong \mathcal{O}_{T^2}(p'' - e)\,. \tag{9.28}$$

Thus far, we have considered points in a smooth fiber T^2. Using the zero section $e = e_9$ of B, we can try to extend our discussion to a family of elliptic curves, that is, to sections of our elliptic fibration $\beta : B \to \mathbb{P}^1$. Consider any section ξ in $\Gamma(B)$. We can, as mentioned above, define a map

$$\xi \to \mathcal{O}_B(\xi)\,. \tag{9.29}$$

This specifies a map from $\Gamma(B) \to Pic(B)$. But similarly to the discussion above, we can, alternatively, use the zero section e to define, for any section ξ, the mapping

$$\xi \to \mathcal{O}_B(\xi - e)\,. \tag{9.30}$$

This map has a similar property to the case of one elliptic curve. That is, for any three sections $\xi, \xi', \xi'' \in \Gamma(B)$ which obey

$$\xi \dot{+} \xi' = \xi'' \tag{9.31}$$

we have

$$\mathcal{O}_B(\xi - e)|_f \otimes \mathcal{O}_B(\xi' - e)|_f = \mathcal{O}_B(\xi'' - e)|_f\,, \tag{9.32}$$

where the equation only holds for the restriction to any smooth fiber f. However, care has to be taken at the singular fibers.

Let us now calculate the class $t_{e_6}(\xi)$ for any section ξ. To do this, we have to calculate the line bundle $\mathcal{O}_B(t_{e_6}^{-1}(\xi))$, as discussed in (9.24). Using the map (9.29), this line bundle corresponds to the section

$$\xi \dot{+} (\dot{-} e_6) = \xi \dot{-} e_6 = t_{e_6}^{-1}(\xi)\,. \tag{9.33}$$

Using equations (9.31) and (9.32), we find that

$$\mathcal{O}_B(\xi - e)|_f \otimes \mathcal{O}_B^{-1}(e_6 - e)|_f = \mathcal{O}_B(t_{e_6}^{-1}(\xi) - e)|_f \tag{9.34}$$

where we have used $\mathcal{O}_B^{-1}(e_6 - e)$ as the second factor since, in this case, $\xi' = -e_6$. Tensoring (9.34) with $\mathcal{O}_B(e)$ gives an equation in terms of divisors, namely

$$t_{e_6}(\xi) = \xi - e_6 + e_9 + a_1 n_1 + a_2 n_2 + a_3 n_3 + a_4 n_4 + a_5 n_5 + a_6 n_6 + a_0 f\,, \tag{9.35}$$

where $a_0, \ldots, a_6$ are integers. Note that this equation holds for both smooth and singular fibers. It even holds without restriction to any fiber at all. We have assured this by adding the fiber components. The specific coefficients $a_0, \ldots, a_6$ will now be determined.

To begin with, intersect both sides of (9.35) with the exceptional divisors n_i for $i = 1, \ldots, 6$. Taking into account the results of table 3, that is, $(t_{e_6})(n_k) = o_k,\ k = 1, \ldots, 4,\ (t_{e_6})(n_k) = n_k,\ k = 5, 6$, the intersection numbers and the fact that $n_i \cdot t_{e_6}(\xi) = t_{e_6}(n_i) \cdot \xi$, we obtain

$$\begin{aligned} o_1 \cdot \xi &= n_1 \cdot \xi + 1 - 2a_1 \\ o_2 \cdot \xi &= n_2 \cdot \xi + 1 - 2a_2 \\ o_3 \cdot \xi &= n_3 \cdot \xi - 1 - 2a_3 \\ o_4 \cdot \xi &= n_4 \cdot \xi - 1 - 2a_4 \\ n_5 \cdot \xi &= n_5 \cdot \xi - 2a_5 \\ n_6 \cdot \xi &= n_6 \cdot \xi - 2a_6\,. \end{aligned} \tag{9.36}$$

	a_1	a_2	a_3	a_4	a_0
e_1	0	0	-1	-1	1
e_2	0	0	-1	-1	1
e_4	0	0	-1	-1	1
e_7	0	1	-1	0	0

Table 4: The coefficients $a_0, \ldots, a_4$ for the expansion of the pull-back of the sections e_1, e_2, e_4 and e_7 under the action of t_{e_6}.

Hence,

$$a_5 = a_6 = 0 \tag{9.37}$$

independently of the choice of section ξ. Recall that e_1, e_2, e_4 and e_7 are sections of B, that is, they intersect each I_2 fiber exactly once. Furthermore, e_1 and e_2 intersect none of the exceptional divisors, the only exceptional divisors which are intersected by e_4 are n_5 and n_6, and e_7 intersects only n_2, n_4 and n_6. Using these facts, we can determine the coefficients $a_1, \ldots, a_4$ for e_1, e_2, e_4 and e_7. The results are presented in the first four columns of table 4. Note that these coefficients are identical for e_1, e_2 and e_4, but differ for e_7.

To determine the remaining coefficients a_0, we use the fact that $t_{e_6}(\xi)$ is again a section and, hence, $t_{e_6}(\xi) \cdot t_{e_6}(\xi) = -1$. Let us begin with e_1, e_2 and e_4. Using (9.35) and the results in table 4, we have

$$t_{e_6}(e_i) = e_i - e_6 + e_9 - n_3 - n_4 + a_0 f \qquad i = 1, 2, 4\,. \tag{9.38}$$

Squaring this expression then tells us that $a_0 = 1$ for e_1, e_2 and e_4. Hence, we have at long last determined the transformation laws

$$t_{e_6}(e_i) = f - l + e_i + e_6 + e_7 + e_9 \qquad i = 1, 2, 4\,, \tag{9.39}$$

where we used the definition of n_3 and n_4 in term of the sections $e_1, \ldots, e_9$ and l given in (6.11). Similarly, using the expansion of $t_{e_6}(e_7)$ determined by (9.35) and table 4, we find that $a_0 = 0$ for e_7. Hence,

$$t_{e_6}(e_7) = l - e_5 - e_8\,. \tag{9.40}$$

JHEP01(2004)059

Our results for the coefficients a_0 are listed in the last column of table 4. We can now use $t_{e_6}(n_5)$ determined in the previous sub-section and (9.38) to show that

$$t_{e_6}(e_3) = t_{e_6}(e_4 + n_5) = f - l + e_4 + e_6 + e_7 + e_9 + n_5 = f - l + e_3 + e_6 + e_7 + e_9\,. \quad (9.41)$$

Finally, using the transformation laws of f and $e_1, \ldots, e_9$ it is easy to read off the transformation of l. We find that

$$t_{e_6}(l) = 2f - e_5 + 2e_6 + e_7 - e_8 + 2e_9\,. \quad (9.42)$$

A general summary of our findings for t_{e_6} is presented in table 6.

9.3 Transformations of the canonical generators under t_{e_4}

The calculation of the transformation laws of the canonical set of generators of $H_2(B, \mathbb{Z})$ under t_{e_4} closely follows the calculation of the transformations under t_{e_6}. First, we determine the transformations of e_9, e_4, e_8 and e_3. Using the definition of a translation, and the facts that $e = e_9$ is the zero section and that e_4 is a section of order two, we find

$$t_{e_4}(e_9) = e_9 \dot{+} e_4 = e_4 \quad (9.43)$$

and

$$t_{e_4}(e_4) = e_4 \dot{+} e_4 = e_9\,. \quad (9.44)$$

Using this, (6.11) and table 3, it follows that

$$t_{e_4}(e_8) = t_{e_4}(e_9 + n_1) = e_4 + o_1 = e_4 + f - n_1 = f + e_4 - e_8 + e_9 \quad (9.45)$$

and

$$t_{e_4}(e_3) = t_{e_4}(e_4 + n_5) = e_9 + o_5 = e_9 + f - n_3 = f - e_3 + e_4 + e_9\,. \quad (9.46)$$

To finish our calculation of the transformation laws of the canonical set of generators of $H_2(B, \mathbb{Z})$ under t_{e_4}, we have to use a formula similar to (9.35) which, for t_{e_4}, is given by

$$t_{e_4}(\xi) = \xi - e_4 + e_9 + a_1 n_1 + a_2 n_2 + a_3 n_3 + a_4 n_4 + a_5 n_5 + a_6 n_6 + a_0 f\,. \quad (9.47)$$

To determine the coefficients for specific sections ξ, we intersect both sides of (9.47) with n_i for $i = 1, \ldots, 6$. Taking into account the results of table 3, that is, $(t_{e_4})(n_k) = o_k$ for $k = 1, 2, 5, 6$ and $(t_{e_4})(n_k) = n_k$ for $k = 3, 4$, we obtain

$$\begin{aligned}
o_1 \cdot \xi &= n_1 \cdot \xi + 1 - 2a_1 \\
o_2 \cdot \xi &= n_2 \cdot \xi + 1 - 2a_2 \\
n_3 \cdot \xi &= n_3 \cdot \xi - 2a_3 \\
n_4 \cdot \xi &= n_4 \cdot \xi - 2a_4 \\
o_5 \cdot \xi &= n_5 \cdot \xi - 1 - 2a_5 \\
o_6 \cdot \xi &= n_6 \cdot \xi - 1 - 2a_6\,.
\end{aligned} \quad (9.48)$$

Hence,

$$a_3 = a_4 = 0 \tag{9.49}$$

independently of the section ξ. Recall that e_1, e_2, e_6 and e_7 are sections, that is, they intersect each I_2 fiber exactly once. Furthermore, e_1 and e_2 intersect none of the exceptional divisors and the only exceptional divisors which are intersected by e_6 are n_3 and n_4. Using these facts, we can determine the coefficients $a_1, \ldots, a_4$ for e_1, e_2, e_6 and e_7. The first four columns of table 5 summarize our findings for the coefficients $a_1, \ldots, a_4$.

	a_1	a_2	a_3	a_4	a_0
e_1	0	0	-1	-1	1
e_2	0	0	-1	-1	1
e_6	0	0	-1	-1	1
e_7	0	1	-1	0	0

Table 5: The coefficients $a_0, \ldots, a_4$ for the expansion of the pull-back of the sections e_1, e_2, e_6 and e_7 under the action of t_{e_4}.

To determine the remaining coefficients a_0, we use the fact that $t_{e_4}(\xi)$ is again a section. Hence $t_{e_4}(\xi) \cdot t_{e_4}(\xi) = -1$. Using this, and squaring the expansion

$$t_{e_4}(e_i) = e_i - e_4 + e_9 - n_5 - n_6 + af \qquad i = 1, 2, 6\,, \tag{9.50}$$

we find $a_0 = 1$ for e_1, e_2 and e_6. Hence,

$$t_{e_4}(e_i) = f - l + e_i + e_4 + e_7 + e_9 \qquad i = 1, 2, 6\,, \tag{9.51}$$

where we used the definitions of n_5 and n_6 given in (6.11). Using the expansion of $t_{e_4}(e_7)$, we find $a_0 = 0$ for e_7 and, hence, that

$$t_{e_4}(e_7) = l - e_3 - e_8\,. \tag{9.52}$$

Our results for the coefficients a_0 are listed in the last column of table 5. We can now show

$$t_{e_4}(e_5) = t_{e_4}(e_6 + n_3) = f - l + e_6 + e_4 + e_7 + e_9 + n_3 = f - l + e_4 + e_5 + e_7 + e_9\,. \tag{9.53}$$

Using the transformation laws of f and $e_1, \ldots, e_9$ it is easy to read of the transformation of l. We find that

$$t_{e_4}(l) = 2f - e_3 + 2e_4 + e_7 - e_8 + 2e_9\,. \tag{9.54}$$

A general summary of our findings for t_{e_4} are presented in table 6.

10. The τ_{B1}, τ_{B2} action on $H_2(B, \mathbb{Z})$ and invariant classes

Having determined the transformation laws of the canonical set of generators $e_1, \ldots, e_9$ and l of $H_2(B, \mathbb{Z})$ under α_B in section 8 and t_{e_4}, t_{e_6}, in section 9, we can now compose them using $\tau_{B1} = t_{e_6} \circ \alpha_B$ and $\tau_{B2} = t_{e_4} \circ \alpha_B$ to determine the action of τ_{B1} and τ_{B2}. The results are presented in the fifth and sixth columns of table 6 respectively. The table also summarizes our results for $(-1)_B$, α_B, t_{e_6} and t_{e_4}.

We now want to find all classes in $H_2(B, \mathbb{Z})$ that are invariant under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group. Let us denote the subspace of all such invariant classes by $H_2(B, \mathbb{Z})^{\text{inv}}$. We will show in [24] that

$$\text{rank} H_2(B, \mathbb{Z})^{\text{inv}} = 4\,. \tag{10.1}$$

	$(-1)_B$	α_B	t_{e_6}	t_{e_4}	τ_{B1}	τ_{B2}
f	f	f	f	f	f	f
e_1	$l-e_7-e_1$	$l-e_7-e_1$	$f-l+e_1+e_6+e_7+e_9$	$f-l+e_1+e_4+e_7+e_9$	$f-e_1+e_6+e_9$	$f-e_1+e_4+e_9$
e_2	$l-e_7-e_2$	e_2	$f-l+e_2+e_6+e_7+e_9$	$f-l+e_2+e_4+e_7+e_9$	$f-l+e_2+e_6+e_7+e_9$	$f-l+e_2+e_4+e_7+e_9$
e_3	e_3	$l-e_3-e_7$	$f-l+e_3+e_6+e_7+e_9$	$f-e_3+e_4+e_9$	$f-e_3+e_6+e_9$	$f-l+e_3+e_4+e_7+e_9$
e_4	e_4	e_4	$f-l+e_4+e_6+e_7+e_9$	e_9	$f-l+e_4+e_6+e_7+e_9$	e_9
e_5	e_5	$l-e_5-e_7$	$f-e_5+e_6+e_9$	$f-l+e_5+e_4+e_7+e_9$	$f-l+e_5+e_6+e_7+e_9$	$f-e_5+e_4+e_9$
e_6	e_6	e_6	e_9	$f-l+e_6+e_4+e_7+e_9$	e_9	$f-l+e_6+e_4+e_7+e_9$
e_7	$l-e_1-e_2$	$2l-(e_1+e_3+e_5+e_7+e_8)$	$l-e_5-e_8$	$l-e_3-e_8$	$l-e_1-e_3$	$l-e_1-e_5$
e_8	e_8	$l-e_7-e_8$	$f+e_6-e_8+e_9$	$f+e_4-e_8+e_9$	$f-l+e_6+e_7+e_8+e_9$	$f-l+e_4+e_7+e_8+e_9$
e_9	e_9	e_9	e_6	e_4	e_6	e_4
l	$2l-(e_1+e_2+e_7)$	$3l-(e_1+e_3+e_5+2e_7+e_8)$	$2f-e_5+2e_6+e_7-e_8+2e_9$	$2f-e_3+2e_4+e_7-e_8+2e_9$	$2f-e_1-e_3+2e_6+e_7+2e_9$	$2f-e_1+2e_4-e_5+e_7+2e_9$

Table 6: The action of $(-1)_B$, α_B, t_{e_6}, t_{e_4}, τ_{B1} and τ_{B2} on $H_2(B,\mathbb{Z})$. Note that f can be expressed as $f = 3l - \sum_{i=1}^{9} e_i$.

Hence, there exist four generators of $H_2(B,\mathbb{Z})^{\text{inv}}$. One of these generators is trivial to find, namely

$$f = 3l - \sum_{i=1}^{9} e_i\,. \tag{10.2}$$

This follows from the first line of table 6, which reflects the statement, shown in [20], that f is preserved by any automorphism of B.

A second invariant generator, consisting only of fiber components of $\beta : B \to \mathbb{P}^1$, is also easy to find. We see using table 3 that

$$n_1 + o_2 \tag{10.3}$$

is preserved under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group. Using (6.11), this can be expressed in terms of the canonical set of generators as

$$n_1 + o_2 = 2l - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + e_8 - e_9\,. \tag{10.4}$$

JHEP01(2004)059

To find the third invariant generator, we exploit the fact that, for any $\xi \in H_2(B,\mathbb{Z})$, the class

$$\xi + \tau_{B1}(\xi) + \tau_{B2}(\xi) + \tau_{B1} \circ \tau_{B2}(\xi) \tag{10.5}$$

is invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$. Choosing $\xi = e_9$, we take as the third generator of $H_2(B,\mathbb{Z})^{\text{inv}}$

$$i \equiv e_9 + \tau_{B1}(e_9) + \tau_{B2}(e_9) + \tau_{B1} \circ \tau_{B2}(e_9)\,. \tag{10.6}$$

Using the results in table 6, we find that

$$i = 2e_6 + 2e_4 - n_1 + o_2\,. \tag{10.7}$$

This can be expressed in terms of the canonical basis as

$$i = 2l - e_1 - e_2 - e_3 + e_4 - e_5 + e_6 - e_8 + e_9\,. \tag{10.8}$$

Finally, we find the fourth invariant generator as a result of enlightened trial and error. The result is

$$M \equiv 2e_2 - 2e_9 - n_1 - n_2\,, \tag{10.9}$$

which, using (6.11), can be written as

$$M = -l + 2e_2 + e_7\,. \tag{10.10}$$

The invariance of M under $\mathbb{Z}_2 \times \mathbb{Z}_2$ is easily checked using the results in table 6. It is straightforward to show that $f, n_1 + o_2, i$ and M are linearly independent classes. We conclude that

$$f\,, \qquad n_1 + o_2\,, \qquad i = 2e_6 + 2e_4 - n_1 + o_2\,, \qquad M = 2e_2 - 2e_9 - n_1 - n_2 \tag{10.11}$$

form a set of generators[2] of $H_2(B,\mathbb{Z})^{\text{inv}}$. Their intersection numbers are displayed in table 7. This choice of generators will turn out to be very convenient when we solve the numerical conditions on the rank four vector bundles V on the Calabi-Yau space X imposed by particle physics phenomenology [22, 24]. Note that it is very easy to find other invariant classes in $H_2(B,\mathbb{Z})$. For example, it is immediately obvious from table 3 that $n_2 + o_1$ is an invariant class. Furthermore, as we stated, (10.5) is an invariant class for any $\xi \in H_2(B,\mathbb{Z})$. However, it follows from (10.1) and the linear independence of $f, n_1 + o_2, i$ and M that

	i	f	$n_1 + o_2$	M
i	-4	4	4	0
f	4	0	0	0
$n_1 + o_2$	4	0	-4	0
M	0	0	0	-4

Table 7: The intersection numbers of the generators of $H_2(B,\mathbb{Z})^{\text{inv}}$.

[2]To be precise, the set $\{f, n_1 + o_2, i, M\}$ generates a sub-lattice of maximal rank in $H_2(B,\mathbb{Z})^{\text{inv}}$. They are, in fact, generators of $H_2(B,\mathbb{Q})^{\text{inv}}$ and not $H_2(B,\mathbb{Z})^{\text{inv}}$. Be that as is may, for our purposes in this paper and in [22, 24], these classes are the most convenient. First of all, since they generate a lattice of the same rank as $H_2(B,\mathbb{Z})^{\text{inv}}$, they demonstrate that $H_2(B,\mathbb{Z})^{\text{inv}}$ is non-empty and of rank four. Secondly, they have convenient intersection numbers which simplify the construction of holomorphic vector bundles in [22, 24]. Since this subtlety does not play a role in either this paper or [22, 24], here and henceforth we refer to them simply as the generators of $H_2(B,\mathbb{Z})^{\text{inv}}$.

any other invariant class must be a linear combination of these four generator classes. For example, one easily finds using (6.11) that

$$n_2 + o_1 = 2f - (n_1 + o_2)\,. \tag{10.12}$$

Finally, we denote the set of generators for invariant curve classes on $B^{'}$ by

$$f^{'}\,, \qquad n_1^{'} + o_2^{'}\,, \qquad i^{'}\,, \qquad M^{'}\,, \tag{10.13}$$

where $i^{'}$ and $M^{'}$ are defined similarly to (10.7) and (10.9).

11. The homology of X and Z

Recall from [20] and section 2 that our analysis involves two different types of Calabi-Yau threefolds. The first are the Calabi-Yau spaces X which are simply connected and admit a freely acting automorphism group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The second are the smooth quotient spaces

$$Z = \frac{X}{(\mathbb{Z}_2 \times \mathbb{Z}_2)} \tag{11.1}$$

with fundamental group $\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_2$. In this section, we will calculate the relevant parts of the homology rings $H_*(X, \mathbb{Z})$ and $H_*(Z, \mathbb{Z})$ of these two types of threefolds. More precisely, we will first determine

$$\dim_{\mathbb{C}} H_i(X, \mathbb{C}) = \dim_{\mathbb{C}}(H_i(X, \mathbb{Z}) \otimes \mathbb{C})\,, \qquad i = 0, \ldots, 6 \tag{11.2}$$

and then show how to find a set of generators of $H_4(X, \mathbb{Z})$. A subset of these generators consists of classes that are invariant under the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$. These span the subgroup of invariant homology classes, which we denote by $H_4(X, \mathbb{Z})^{\mathrm{inv}} \subset H_4(X, \mathbb{Z})$. We will explicitly compute a complete set of invariant generators. Using these results, we then determine

$$\dim_{\mathbb{C}} H_i(Z, \mathbb{C}) = \dim_{\mathbb{C}}(H_i(Z, \mathbb{Z}) \otimes \mathbb{C})\,, \qquad i = 0, \ldots, 6 \tag{11.3}$$

and explicitly find a set of generators of $H_4(Z, \mathbb{Z})$. Both quantities, the dimensions of $H_i(Z, \mathbb{C})$, $i = 0, \ldots, 6$ and the generators of $H_4(Z, \mathbb{Z})$, are of physical importance. The dimension of the homology groups of Z are a vital ingredient in determining the particle spectrum of string theories compactified on the Calabi-Yau threefold Z. The elements of $H_4(Z, \mathbb{Z})$ are divisors in Z which clearly descend, via the modding out (11.1), from the invariant divisors on X, that is, the elements of $H_4(X, \mathbb{Z})^{\mathrm{inv}}$. The classes in $H_4(X, \mathbb{Z})^{\mathrm{inv}}$ will be used in [21] and [24] to construct invariant stable vector bundles on X which, in turn, descend to stable vector bundles on Z. These bundles admit connections which correspond to solutions of the hermitian Yang-Mills equations on Z.

Let us begin with the Calabi-Yau threefold X. It is convenient to use Poincaré duality, that is

$$H_i(X, \mathbb{Z}) \cong H^{6-i}(X, \mathbb{Z})\,, \qquad 0 = 1, \ldots, 6\,. \tag{11.4}$$

Furthermore, the fact that X is a Kahler manifold allows us to write the Hodge diamond of X as in figure 11. Here

$$h_X^{a,b} = \dim_{\mathbb{C}} H_{\bar{\partial}}^{a,b}(X)\,, \qquad a,b = 0,\ldots,3 \tag{11.5}$$

denotes the dimension of the a-th Dolbeaux cohomology of holomorphic b-forms on X. It is well-known that for Kahler manifolds

$$H^k(X,\mathbb{C}) = \sum_{a+b=k} H_{\bar{\partial}}^{a,b}(X) \tag{11.6}$$

and, therefore, that

$$\dim_{\mathbb{C}} H^k(X,\mathbb{C}) = \sum_{a+b=k} h^{a,b}\,. \tag{11.7}$$

For simply connected Calabi-Yau threefolds, the Hodge diamond simplifies to the highly symmetric form given in figure 12. Hence, to determine the dimension of the homology groups of X, we need to determine $h_X^{1,1}$ and $h_X^{2,1}$ only. To do this, we begin by calculating the elements of $H_{\bar{\partial}}^{1,1}(X)$ explicitly. Since X is simply connected, we have

$$H_{\bar{\partial}}^{1,1}(X) \cong Pic(X) \otimes \mathbb{C} \cong H^2(X,\mathbb{Z}) \otimes \mathbb{C}\,. \tag{11.8}$$

Therefore, if we can determine the generators of all line bundles on X, we will have determined all generators of $H^2(X,\mathbb{Z})$ and, hence, of $H_{\bar{\partial}}^{1,1}(X)$. Recall that X is the fiber product of two rational elliptic surfaces B and B'. Hence, we have the inclusion map

$$i : B \times_{\mathbb{P}^1} B' \to B \times B'\,. \tag{11.9}$$

In terms of cohomology, this defines the pull-back map

$$i^* : H^2(B \times B',\mathbb{Z}) \to H^2(B \times_{\mathbb{P}^1} B',\mathbb{Z})\,. \tag{11.10}$$

Using the Kunneth formula, we find

$$H^2(B \times B',\mathbb{Z}) \cong H^2(B,\mathbb{Z}) \oplus H^2(B',\mathbb{Z})\,. \tag{11.11}$$

With the help of the the isomorphisms

$$H^2(B \times_{\mathbb{P}^1} B',\mathbb{Z}) \cong Pic(B \times_{\mathbb{P}^1} B')\,, \qquad H^2(B,\mathbb{Z}) \cong Pic(B)\,, \qquad H^2(B',\mathbb{Z}) \cong Pic(B')\,, \tag{11.12}$$

we have

$$i^* : Pic(B) \oplus Pic(B') \to Pic(B \times_{\mathbb{P}^1} B')\,. \tag{11.13}$$

To see how the map i^* works, recall that for X we had the natural projections

$$\begin{tikzcd} & X \arrow[dl, "\pi'"'] \arrow[dr, "\pi"] & \\ B \arrow[dr, "\beta"'] & & B' \arrow[dl, "\beta'"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{11.14}$$

$$
\begin{array}{ccccccc}
 & & & h_X^{0,0} & & & \\
 & & h_X^{1,0} & & h_X^{0,1} & & \\
 & h_X^{2,0} & & h_X^{1,1} & & h_X^{0,2} & \\
h_X^{3,0} & & h_X^{2,1} & & h_X^{1,2} & & h_X^{0,3} \\
 & h_X^{3,1} & & h_X^{2,2} & & h_X^{1,3} & \\
 & & h_X^{3,2} & & h_X^{2,3} & & \\
 & & & h_X^{3,3} & & &
\end{array}
$$

Figure 11: The Hodge diamond of the Kahler manifold X.

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h_X^{1,1} & & 0 & \\
1 & & h_X^{2,1} & & h_X^{2,1} & & 1 \\
 & 0 & & h_X^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 12: The Hodge diamond of the Calabi-Yau threefold X.

Consider any line bundle L on B determined by the divisor D. That is,

$$L = \mathcal{O}_B(D)\,. \tag{11.15}$$

Then, clearly

$$\pi^{'*}L = \pi^{'*}\mathcal{O}_B(D) = \mathcal{O}_X(\pi^{'-1}(D)) = \mathcal{O}_X(D \times_{\mathbb{P}^1} B^{'})\,. \tag{11.16}$$

Similarly, for a line bundle $L^{'}$ on $B^{'}$ determined by the divisor $D^{'}$, that is

$$L^{'} = \mathcal{O}_{B^{'}}(D^{'})\,, \tag{11.17}$$

we obtain

$$\pi^{*}L^{'} = \pi^{*}\mathcal{O}_{B^{'}}(D^{'}) = \mathcal{O}_X(\pi^{-1}(D^{'})) = \mathcal{O}_X(B \times_{\mathbb{P}^1} D^{'})\,. \tag{11.18}$$

Hence, the map (11.13) is given by

$$i^{*} : (L, L^{'}) \to \pi^{'*}L \otimes \pi^{*}L^{'}\,. \tag{11.19}$$

It is not to difficult to show that the map i^* is surjective, that is, all line bundles on $B \times_{\mathbb{P}^1} B^{'}$ arise from line bundles on B and $B^{'}$ via (11.19). But what is the kernel of i^*? It follows from (11.19) that $(L, L^{'})$ will be in the kernel of i^* if and only if

$$\pi^{'*}L = (\pi^{*}L^{'})^{-1}\,. \tag{11.20}$$

Hence, the divisors $\pi^{'-1}(D)$ and $\pi^{-1}(D^{'})$ associated with $\pi^{'*}L$ and $\pi^{*}L^{'}$ respectively must satisfy

$$\pi^{'-1}(D) = -\pi^{-1}(D^{'})\,. \tag{11.21}$$

This can only happen if L and $L^{'}$ are pull-back bundles from $\mathbb{P}^1$ of the form

$$L = \beta^{*}\mathcal{O}_{\mathbb{P}^1}(a)\,, \qquad L^{'} = \beta^{'*}\mathcal{O}_{\mathbb{P}^1}(-a)\,, \qquad a \in \mathbb{Z}\,. \tag{11.22}$$

Then

$$\pi^{'*}L = \mathcal{O}_X(a(f \times f^{'}))\,, \qquad \pi^{*}L^{'} = \mathcal{O}_X(-a(f \times f^{'})) \tag{11.23}$$

and, hence, (11.21) is satisfied. Noting that $\mathcal{O}_{\mathbb{P}^1}(a)$, $a \in \mathbb{Z}$ spans $Pic(\mathbb{P}^1) \cong H^2(\mathbb{P}^1, \mathbb{Z})$, it follows that

$$H^2(X, \mathbb{Z}) \cong Pic(B \times_{\mathbb{P}^1} B^{'}) \cong \frac{Pic(B) \oplus Pic(B^{'})}{Pic(\mathbb{P}^1)} \cong \frac{H^2(B, \mathbb{Z}) \oplus H^2(B^{'}, \mathbb{Z})}{H^2(\mathbb{P}^1, \mathbb{Z})}\,. \tag{11.24}$$

This result allows us to determine all of the relevant quantities on X, namely, the Hodge numbers $h^{1,1}_X, h^{2,1}_X$ and the generators of $H_4(X, \mathbb{Z})$.

First consider the Hodge numbers. Using (11.5), (11.8), (11.24) and the fact that $\text{rank}H^2(B, \mathbb{Z}) = 10$, we find

$$h^{1,1}_X = \dim_{\mathbb{C}} H^{1,1}_{\bar{\partial}}(X) = \dim_{\mathbb{C}} H^2(X, \mathbb{C}) = 10 + 10 - 1 = 19\,. \tag{11.25}$$

To determine $h^{2,1}_X$, note that for complex threefolds the topological Euler characteristic can be calculated as

$$\chi(X) = \sum_{i=0}^{6} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathbb{C})\,. \tag{11.26}$$

Furthermore, it can be shown that

$$\chi(X) = c_3(X)\,. \tag{11.27}$$

But, as discussed in [20], $c_3(X) = 0$. Using this and the Hodge diamond of figure 12, we find

$$0 = \chi(X) = 1 + h_X^{1,1} - 1 - 2h_X^{2,1} - 1 + h_X^{1,1} + 1\,. \tag{11.28}$$

It then follows, using (11.25), that

$$h^{2,1} = 19\,. \tag{11.29}$$

Note that using (11.7), the Hodge diamond of figure 12 and $h_X^{1,1} = h_X^{2,1} = 19$ we can determine $\dim H^i(X, \mathbb{C})$ and, hence, by Poincaré duality $\dim H_i(X, \mathbb{C})$ for all $i = 0, \dots, 6$.

Now, recall from Poincaré duality (11.2) that

$$H_4(X, \mathbb{Z}) \cong H^2(X, \mathbb{Z})\,. \tag{11.30}$$

Using this and (11.24), we can determine the nineteen generators of $H_4(X, \mathbb{Z})$. First, note that if D is a divisor in B, that is, $D \in H_2(B, \mathbb{Z})$, then

$$\pi'^{-1}(D) = D \times_{\mathbb{P}^1} B' \in H_4(X, \mathbb{Z})\,. \tag{11.31}$$

Similarly, if $D' \in H_2(B', \mathbb{Z})$ then

$$\pi^{-1}(D') = B \times_{\mathbb{P}^1} D' \in H_4(X, \mathbb{Z})\,. \tag{11.32}$$

Taking $\{D\}$ to be the canonical set of generators $\{e_1, \dots, e_9, l\}$ of $H_2(B, \mathbb{Z})$ and $\{D'\}$ to be $\{e'_1, \dots, e'_9, l'\}$ in $H_2(B', \mathbb{Z})$, we see from (11.24) that the set

$$\{D\} \times_{\mathbb{P}^1} B'\,, \qquad B \times_{\mathbb{P}^1} \{D'\} \tag{11.33}$$

generates $H_4(X, \mathbb{Z})$. Note that there are only nineteen, and not twenty, classes in (11.33) since

$$f \times_{\mathbb{P}^1} B' = B \times_{\mathbb{P}^1} f' = f \times f'\,. \tag{11.34}$$

This is the reason for the $H^2(\mathbb{P}^1, \mathbb{Z})$ denominator in (11.24). We conclude that (11.33), subject to (11.34), gives an explicit set of nineteen generators of $H_4(X, \mathbb{Z})$.

So far, we have analyzed the Hodge diamond and fourth homology group of the Calabi-Yau threefold X. We now turn our attention to the Calabi-Yau threefold Z defined in (11.1). Here, given the previous results, it is easiest to proceed in reverse, first determining the generators of $H_4(Z, \mathbb{Z})$ and then the Hodge numbers. To to this, note that any class in $H_4(Z, \mathbb{Z})$ must descend under modding out from a $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant class in $H_4(X, \mathbb{Z})$. If we denote the subspace of invariant classes by $H_4(X, \mathbb{Z})^{\text{inv}} \subset H_4(X, \mathbb{Z})$, then there exists an isomorphism

$$H_4(X, \mathbb{Z})^{\text{inv}} \cong H_4(Z, \mathbb{Z})\,. \tag{11.35}$$

What are the generators of $H_4(X, \mathbb{Z})^{\text{inv}}$? Recall from (2.6) that the generators of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism group on X are

$$\tau_{Xi} = \tau_{Bi} \times_{\mathbb{P}^1} \tau_{B'i}\,, \qquad i = 1, 2\,. \tag{11.36}$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 7 & & 0 & \\
1 & & 7 & & 7 & & 1 \\
 & 0 & & 7 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\quad .
$$

Figure 13: The Hodge diamond of the Calabi-Yau threefold Z.

Acting with these involutions on (11.31) and (11.32) we have

$$\tau_{Xi}(D \times_{\mathbb{P}^1} B^{'}) = \tau_{Bi}(D) \times_{\mathbb{P}^1} B^{'} \tag{11.37}$$

and

$$\tau_{Xi}(B \times_{\mathbb{P}^1} D^{'}) = B \times_{\mathbb{P}^1} \tau_{B^{'}i}(D^{'}) \tag{11.38}$$

respectively for $i = 1, 2$. It follows from (11.37) that $D \times_{\mathbb{P}^1} B^{'}$ will be in $H_4(X, \mathbb{Z})^{\mathrm{inv}}$ if and only if

$$\tau_{Bi}(D) = D\,, \qquad i = 1, 2\,. \tag{11.39}$$

That is, if and only if

$$D \in H_2(B, \mathbb{Z})^{\mathrm{inv}}\,. \tag{11.40}$$

From the results of the previous section, we know that the classes $f, n_1 + o_2, i$ and M given in (10.11) generate $H_2(B, \mathbb{Z})^{\mathrm{inv}}$. Similarly, it follows from (11.38) that $B \times_{\mathbb{P}^1} D^{'}$ will be in $H_4(X, \mathbb{Z})^{\mathrm{inv}}$ if and only if

$$\tau_{B^{'}i}(D^{'}) = D^{'}\,, \qquad i = 1, 2\,. \tag{11.41}$$

That is

$$D^{'} \in H_2(B^{'}, \mathbb{Z})^{\mathrm{inv}}\,, \tag{11.42}$$

which is generated by the classes $f^{'}, n_1^{'} + o_2^{'}, i^{'}$ and $M^{'}$ given in (10.13). It is clear, then, that a set of generators of $H_4(X, \mathbb{Z})^{\mathrm{inv}}$ is given by

$$\{D_{\mathrm{inv}}\} \times_{\mathbb{P}^1} B^{'}\,, B \times_{\mathbb{P}^1} \{D^{'}_{\mathrm{inv}}\}\,, \tag{11.43}$$

where

$$\{D_{\mathrm{inv}}\} = \{f, n_1 + o_2, i, M\} \tag{11.44}$$

and

$$\{D'_{\rm inv}\} = \{f', n'_1 + o'_2, i', M'\}\,. \tag{11.45}$$

Note that it follows from (11.34) that there are only seven, not eight, classes in (11.43). Another convenient way of writing the generators (11.43) is as

$$\pi'^{-1}(f)\,,\quad \pi'^{-1}(n_1+o_2)\,,\quad \pi'^{-1}(i)\,,\quad \pi'^{-1}(M)\,,\quad \pi^{-1}(n'_1+o'_2)\,,\quad \pi^{-1}(i')\,,\quad \pi^{-1}(M')\,. \tag{11.46}$$

These classes all descend under the modding out of X by $\mathbb{Z}_2 \times \mathbb{Z}_2$ to form a set of seven generators of $H_4(Z,\mathbb{Z})$. We conclude that

$$\mathrm{rank} H_4(Z,\mathbb{Z}) = 7 \tag{11.47}$$

and, therefore, that Z admits the homology classes required to produce anomaly free, three family particle physics theories. From this result, it is straightforward to compute the Hodge numbers of Z. First, note that

$$H^{1,1}_{\bar\partial}(Z) \cong H^2(Z,\mathbb{Z}) \otimes \mathbb{C} \cong H_4(Z,\mathbb{Z}) \otimes \mathbb{C}. \tag{11.48}$$

Then,

$$h^{1,1}_Z = \dim_{\mathbb{C}} H^{1,1}_{\bar\partial}(Z) = \dim_{\mathbb{C}} H_4(Z,\mathbb{C}) = \mathrm{rank} H_4(Z,\mathbb{Z})^{\rm inv} = 7\,. \tag{11.49}$$

The remaining Hodge number, $h^{2,1}_Z$, can be computed using

$$\chi(Z) = \sum_{i=0}^{6} (-1)^i \dim_{\mathbb{C}} H^i(Z,\mathbb{C})\,, \tag{11.50}$$

the fact that

$$\chi(Z) = c_3(Z) = \frac{c_3(X)}{4} = 0 \tag{11.51}$$

and the structure of the Hodge diamond. We find that

$$0 = 1 + h^{1,1}_Z - 1 - 2h^{2,1}_Z - 1 + h^{1,1}_Z + 1 \tag{11.52}$$

which, using (11.49), implies

$$h^{2,1}_Z = 7\,. \tag{11.53}$$

Therefore, the Hodge diamond of Z has the form given in figure 13. Using this, Poincaré duality and (11.7) it is now easy to read off the dimensions of $H_i(Z,\mathbb{Z})$ for $i = 0, \ldots, 6$.

Acknowledgments

René Reinbacher and Burt Ovrut are supported in part by DOE under contract No. DE-AC02-76-ER-03071 and the NSF Focused Research Grant DMS 0139799. In addition, René Reinbacher wishes to acknowledge partial support from an SAS Dissertation Fellowship and a Daimler-Benz Fellowship. Tony Pantev is supported in part by NSF grants DMS 0099715 and DMS 0139799 and an A.P. Sloan Research Fellowship.

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JHEP01(2004)059

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